VIBRATION ANALYSIS OF AN AXIAL-LOADED EULER-BERNOULLI BEAM ON TWO-PARAMETER FOUNDATION

Conference Paper · November 2019

Rômulo Cortez

Simone dos Santos
Universidade Federal do Piauí

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Lucas Oliveira Siqueira
Rômulo Luz Cortez
Simone dos Santos Hoefel
Laboratório de Mecânica Computacional - LaMeC, Universidade Federal do Piauí - UFPI. Teresina-PI, Brazil.
lucasloliveira13@gmail.com, romulocortez10@gmail.com and simone.santos@ufpi.edu.br

Abstract. *The studies of the Euler-Bernoulli beam on an elastic foundation is the basis for analysis of soil-structure interaction. In this paper is analyzed the free vibration of an Euler-Bernoulli beam resting on two-parameter foundation subject to an axial load. The frequency equations is obtained for several boundary conditions. A finite element is developed by mean of the Rayleigh-Ritz method using a cubic approximated polynomial. Numerical comparisons with others works are done for check the method accuracy for this kind of problem. The frequency parameters of the system and the mode shapes are obtained for classical and non-classical boundary conditions. It's verified that the axial load decrease the frequency parameter and the foundation increase the frequency parameter.*

Keywords: *Finite Element Method, Elastic Foundation, Frequency Equations, Free Vibration Analysis, Buckling Analysis.*

1. INTRODUCTION

The study of beams on an elastic foundation with different configurations has great importance for many applications in engineering, once practical problems can be modeled by means of beam-foundation systems. Thus, the analysis of soil-structure interaction have often the foundation formulation coupled to the classical theory of beams such as Euler-Bernoulli and Timoshenko. Applications of soil-structure interaction problems include railroads, road pavements, continuously supported ducts and strip foundations. (Yokoyama, 1996).

Finite element method (FEM) was largely used to solve these types of problems. Franciosi and Masi (1993) used the finite element method to analyze the free vibration of beams on two-parameters foundations and check the performance of the method comparing with the exact solution. De Rosa and Maurizi (1998) determined the natural frequencies of a Euler-Bernoulli beam resting on Pasternak foundation with constraints elastic ends and a concentrated mass at an arbitrary point. Morfidis and Avramidis (2002) developed a new generalized finite element to problems of reinforced concrete or steel structures. Kim and Kim (2012) studied the Euler-Bernoulli beam on Winkler elastic foundation through of the finite element method and comparing with the exact solution. Bezerra *et al.* (2017) analyzed the frequency parameters of a Euler-Bernoulli beam on Pasternak foundation.

The objective of this work is to determine the frequency equations and analyze the effect of the axial load on the dynamic response of an axially loaded Euler-Bernoulli beam supported on an elastic foundation. The FEM is developed and compared to the exact solution. Several numerical examples show the influence of the axial load and the foundation type on the free vibration of an Euler-Bernoulli beam for classical and non-classical boundary conditions.

2. CLASSIC THEORY

Figure 1 shows scheme of a Euler-Bernoulli beam of length L, resting on two-parameter foundation subject to a axial load p. The parameters of the foundation given by k_1 and k_2, that are the elastic linear springs coefficient and the shear layer stiffness coefficient, respectively. The motion equation of the beam can be founded through the energy conservation.

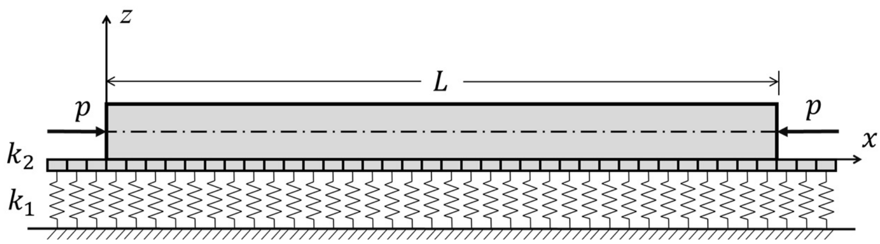

Figure 1. Euler-Bernoulli beam resting on two-parameter foundation subject to a axial load

The strain energy of the system is given by the sum of the parcels strain energy of the beam, strain energy of the elastic foundation, strain energy of the shear and the strain energy caused by the axial load.

$$U = \frac{1}{2}\int_0^L EI\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2 dx + \frac{1}{2}\int_0^L k_1 w^2 dx + \frac{1}{2}\int_0^L k_2\Big(\frac{\partial w}{\partial x}\Big)^2 dx + \frac{1}{2}\int_0^L p\Big(\frac{\partial w}{\partial x}\Big)^2 dx, \tag{1}$$

where I is the moment of inertia of the cross section about the y axis, E the Young's modulus and w the transverse displacement dependent of the axial location x and of the time t.

The kinetic energy of the system can be expressed as:

$$T = \frac{1}{2}\int_0^L \rho A\Big(\frac{\partial w}{\partial t}\Big)^2 dx, \tag{2}$$

in which ρ is the mass density and A is the cross-section area.

The equation of motion can be obtained using Hamilton's principle generalized:

$$\int_{t_1}^{t_2} \delta(T - U + W)dt = 0, \tag{3}$$

where W is the work due non conservative forces, t_1 and t_2 are times at which the configuration of the system is known and δ is the variation operator (Azevedo *et al.*, 2016). Substituting Eqs. (1) and (2) into Eq. (3), after some manipulations, one can obtain the differential equation of motion for the transverse vibration of the beam for free vibration response:

$$EI\Big(\frac{\partial^4 w}{\partial x^4}\Big) + \rho A\Big(\frac{\partial^2 w}{\partial t^2}\Big) + k_1 w - k_2\Big(\frac{\partial^2 w}{\partial x^2}\Big) + p\Big(\frac{\partial^2 w}{\partial x^2}\Big) = 0. \tag{4}$$

Assuming that the beam is excited harmonically and adding non-dimensional parameters:

$$w(x,t) = W(x)e^{i\omega_n t},\ \xi = x/L \text{ and } b^2 = \frac{\rho A \omega_n^2 L^4}{EI}, \tag{5}$$

so that, $i = \sqrt{1}$, ξ is the non-dimensional length of the beam, $W(x)$ is the normal function of the displacement $w(x,t)$ and b is the frequency parameter. Substituting the relations introduced in Eq. (5) into Eq. (4) and omitting the common term $e^{i\omega_n t}$, the following equation is obtained:

$$\frac{d^4 W(\xi)}{d\xi^4} + (P - K_2)\frac{d^2 W(\xi)}{d\xi^2} + (K_1 - b^2)W(\xi) = 0, \tag{6}$$

where K_1 and K_2 are the coefficients related with the elastic and the shear layer stiffness, and P is the load parameter. The parameters are defined as Abohadima *et al.* (2015) and given by, respectively.

$$K_1 = \frac{k_1 L^4}{EI},\ K_2 = \frac{k_2 L^2}{EI}\ and\ P = \frac{pL^2}{EI}. \tag{7}$$

Assuming the solution of the type $W(\xi) = Ce^{s\xi}$, substituting in the equation (6), is obtained the characteristic equation.

$$s^4 + (P - K_2)s^2 + (K_1 - b^2) = 0. \tag{8}$$

Solving the O.D.E (Ordinary Differential Equation), the solution is given by hyperbolic and trigonometric terms as:

$$W(\xi) = C_1 cosh(\alpha\xi) + C_2 sinh(\alpha\xi) + C_3 cos(\beta\xi) + C_4 sin(\beta\xi), \tag{9}$$

where the coefficients α and β are given by:

$$\alpha = \sqrt{\frac{-(P - K_2) + \sqrt{(P - K_2)^2 - 4(K_1 - b^2)}}{2}}, \tag{10}$$

$$\beta = \sqrt{\frac{(P - K_2) + \sqrt{(P - K_2)^2 - 4(K_1 - b^2)}}{2}}. \tag{11}$$

The terms in Eq. (9) C_1, C_2, C_3 and C_4 are constants. Which can be determined from the boundary conditions of the beam.

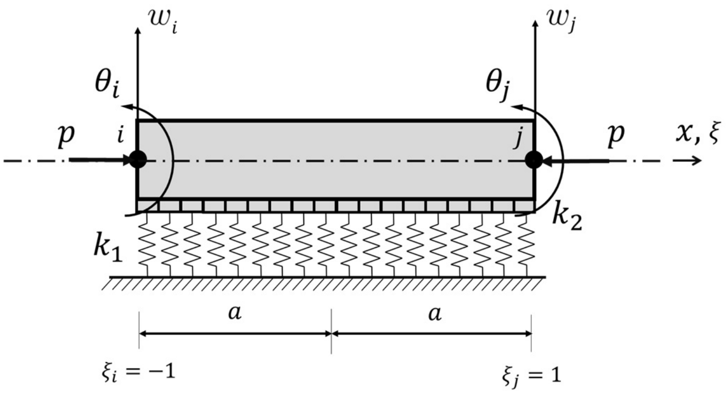

Figure 2. Beam element

3. FINITE ELEMENT METHOD

Consider a uniform axial-loaded Euler-Bernoulli beam element on two-parameter foundation as shown in Figure 2. The beam element consists of two nodes and each node has two degrees of freedom: w, the total deflection, and θ, the slope due to bending (Bezerra *et al.*, 2017). Applying the Rayleigh-Ritz method, the displacement w is approximated for a polynomial function with four constants, so that a_i are constants.

$$w(\xi) = \sum_{i=0}^{3} a_i \xi^i = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3, \tag{12}$$

Using the non-dimensional coordinate ξ, the element length $2a$, and applying the boundary conditions on the points $\xi = -1$ and $\xi = 1$ in the Eq. (12). The matrix form of the displacement can be written as: $w = [\mathbf{N}(\xi)]\{\mathbf{v}\}_e$, where $[\mathbf{N}(\xi)]$ are the shape functions and $\{\mathbf{v}\}_e$ is the vector of nodal coordinates. The subscript e represents expressions for a single element. Therefore, the shape functions and the vector of nodal coordinates, can be expressed as:

$$[\mathbf{N}(\xi)]^T = \frac{1}{4}\begin{bmatrix} 2-3\xi+\xi^3 \\ a(1-\xi-\xi^2+\xi^3) \\ 2+3\xi-\xi^3 \\ a(-1-\xi+\xi^2+\xi^3) \end{bmatrix}, \{\mathbf{v}\}_e^T = \begin{bmatrix} w_1 & \theta_1 & w_2 & \theta_2 \end{bmatrix}. \tag{13}$$

Thus, considering the foundation, the axial load and the beam, the strain and kinetic energy for an element length $2a$ are given by:

$$\mathbf{U}_e = \frac{1}{2}\frac{EI}{a}\int_{-1}^{1}\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 d\xi + \frac{1}{2}k_1 a\int_{-1}^{1} w^2 d\xi + \frac{1}{2}\frac{k_2}{a}\int_{-1}^{1}\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi + \frac{1}{2}\frac{p}{a}\int_{-1}^{1}\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi, \tag{14}$$

$$\mathbf{T}_e = \frac{1}{2}\rho A a\int_{-1}^{1}\left(\frac{\partial w}{\partial t}\right)^2 d\xi. \tag{15}$$

Substituting the displacement expression, Eq. (13), into the strain energy, Eq. (14), gives:

$$\mathbf{U}_e = \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{EI}{a}\int_{-1}^{1}[\mathbf{N}(\xi)'']^T[\mathbf{N}(\xi)'']d\xi + k_1 a\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi + \right.$$
$$\left.\frac{k_2}{a}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']d\xi + \frac{p}{a}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']d\xi\right]\{\mathbf{v}\}_e, \tag{16}$$

where $[\mathrm{N}(\xi)'] = \dfrac{\partial \mathbf{N}(\xi)}{\partial \xi}$. Therefore, the elementary stiffness matrix is given by:

$$[\mathbf{k}]_e = \frac{EI}{a}\int_{-1}^{1}[\mathbf{N}(\xi)'']^T[\mathbf{N}(\xi)'']d\xi + k_1 a\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi +$$
$$\frac{k_2}{a}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']d\xi + \frac{p}{a}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']d\xi, \tag{17}$$

Substituting the displacement expression, Eq. (13), into the kinetic energy, Eq. (15), gives:

$$\mathbf{T}_e = \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T \rho A a \int_{-1}^{1} [\mathbf{N}(\xi)]^T [\mathbf{N}(\xi)] d\xi \{\dot{\mathbf{v}}\}_e\,, \tag{18}$$

hence, the elementary inertia matrix is given by:

$$[\mathbf{m}]_e = \rho A a \int_{-1}^{1} [\mathbf{N}(\xi)]^T [\mathbf{N}(\xi)] d\xi. \tag{19}$$

4. NUMERICAL RESULTS

The frequency parameter was calculated using the element finite method (FEM) to the first three modes and comparing with the exact solution presented in Yokoyama (1996). For the comparison was varied the parameter P, K_1 and K_2 how indicated in the table 1. Thus, consider a Euler-Bernoulli beam with hinged-hinged ends, such that $E/G = 2.5$, $\rho = 2500 kg/m^3$, $r_g = 0.05m$, $I = 0.05^2 m^4$, $A = 1m^2$, $\nu = 0.25$ and $L = 0.5m$. The results are agree with Yokoyama, varying only in the third decimal place.

Table 1. Values of frequency parameters of the first three vibration modes.

Parameters			b_1			b_2			b_3		
P	K_1	K_2	10 ele.	20 ele.	Exact$^{(1)}$	10 ele.	20 ele.	Exact$^{(1)}$	10 ele.	20 ele.	Exact$^{(1)}$
0	0	0	9.87	9.87	9.87	39.48	39.48	39.48	88.87	88.83	88.83
0.6	0	0	6.24	6.24	6.24	36.4	36.4	36.4	85.86	85.82	85.81
0.6	$0.6\pi^4$	0	9.87	9.87	9.87	37.2	37.19	37.19	86.2	86.15	86.15
0.6	$0.6\pi^4$	π^2	13.96	13.96	13.96$^{(2)}$	42.11	42.11	42.25$^{(2)}$	91.15	91.1	92.69$^{(2)}$

$^{(1)}$ (Yokoyama, 1996)
$^{(2)}$ (Results of Yokoyama using FEM.)

It's noticed that for higher modes, increasing the number of elements the difference between the FEM and analytic solutions decreases. Furthermore, it's showed that the addition of an axial compressive load causes a decrease of the frequency parameter, this is due to the increase of the shear effect caused by the contribution of the vertical component of the axial load. Then, adding the Winkler foundation, the frequency parameter increases, however presents lower values than of the unloaded Euler-Bernoulli beam, which is due to the presence of axial force. Finally, for the foundation of two parameters, the frequency parameters increase because the foundation increases the stiffness of the system.

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