DYNAMIC BEHAVIOR OF AN AXIAL-LOADED TIMOSHENKO BEAM ON THE ELASTIC FOUNDATION

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Abstract. *Timoshenko beam is one of the most complete beam models, so studies in this field are of great importance for the practical engineering. In this paper is analyzed the influence of the elastic foundation and the axial load on the dynamic behavior of the Timoshenko beam with uniform cross-section. A Finite Element is developed through of a cubic approximated polynomial. Numerical comparisons and examples are done for check the method performance and present a good agreement. The natural frequencies are obtained for several boundary conditions. It's investigated the slenderness ratio, the axial load parameter and the foundation stiffness influence in the frequency parameter. Finally, the results shows that the axial load decrease the frequency parameter and the foundation increase the frequency parameter.*

Keywords: *Timoshenko Beam Theory, Elastic Foundation, Finite Element Method, Buckling Analysis.*

1. INTRODUCTION

Model proposed by Winkler (1867) is the basis of soil-foundation interaction studies, also known as one-parameter model, where the soil is modeled by elastic linear springs and it is assumed that the soil displacement at the surface is directly proportional to the stress applied at the point. However, the problem associated with this approach is that the displacement occurs only where the load is applied, ie, outside of the loaded region it is zero.

So, various researchers proposed models to correct the shortcomings of the Winkler model. For instance, Filonenko-Borodich (1940) who proposed the first two-parameter model, followed by several researchers such as Hetenyi (1946), Pasternak (1954), Reissner (1958), Kerr (1965) and Vlasov and Leontiev (1966). Further, Yokoyama (1996) used FEM to analyze a Timoshenko beam resting on Pasternak foundation subject to an axial load for several constraints ends. Abohadima *et al.* (2015) investigated the free and forced vibration of an axial-loaded Timoshenko beam resting on a elastic foundation using the Recursive Differentiation Method (RDM).

In this context, the purpose of these article is investigate the dynamic behavior of an axial-loaded Timoshenko beam resting on a two-parameter elastic foundation. FEM is developed and compared with the analytic solution. Foundation parameters and the axial load contributions are investigated in several numerical examples for beams with different end restraints.

2. AXIAL-LOADED BEAM ON ELASTIC FOUNDATION FORMULATION

Differential Equations (1) and (2) express the motion for a freely vibration beam, accordingly to the Timoshenko formulation taking into account the foundation with elastic parameter k_w and shear parameter G_p and the axial load P contributions, as show in Fig. 1 (Wang and Stephens, 1977; Wu, 2013).

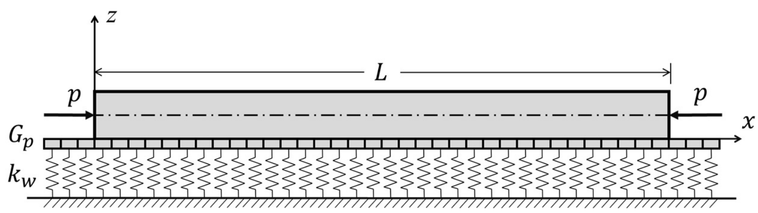

Figure 1. Sketch of an axial-loaded beam resting on a Pasternak foundation

$$EI\left(1+\frac{G_p-P}{\kappa GA}\right)\frac{\partial^4 w(x,t)}{\partial x^4}+\left(\rho A+\frac{\rho I k_w}{\kappa GA}\right)\frac{\partial^2 w(x,t)}{\partial t^2}-\left(\frac{EIk_w}{\kappa GA}+G_p-P\right)\frac{\partial^2 w(x,t)}{\partial x^2}-$$

$$\rho I\left(1+\frac{E}{\kappa G}+\frac{G_p-P}{\kappa GA}\right)\frac{\partial^4 w(x,t)}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 w(x,t)}{\partial t^4}+k_w w(x,t)=0, \tag{1}$$

$$EI\left(1+\frac{G_p-P}{\kappa GA}\right)\frac{\partial^4\psi(x,t)}{\partial x^4}+\left(\rho A+\frac{\rho I k_w}{\kappa GA}\right)\frac{\partial^2\psi(x,t)}{\partial t^2}-\left(\frac{EIk_w}{\kappa GA}+G_p-P\right)\frac{\partial^2\psi(x,t)}{\partial x^2}-$$
$$\rho I\left(1+\frac{E}{\kappa G}+\frac{G_p-P}{\kappa GA}\right)\frac{\partial^4\psi(x,t)}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4\psi(x,t)}{\partial t^4}+k_w\psi(x,t)=0 \tag{2}$$

in these equations, E is the Young's modulus, I is the moment of inertia, ρ is the mass density, A is the cross-section area, t is the time, κ is the shear factor, G is the shear modulus, $w(x,t)$ is the vertical displacement and $\psi(x,t)$ is the bending rotation.

Assuming that the beam is excited harmonically one has

$$w(x,t)=W(x)e^{i\omega_n t},\ \psi(x,t)=\Psi(x)e^{i\omega_n t} \text{ and } \xi=x/L \tag{3}$$

where $W(x)$ and $\Psi(x)$ are the amplitudes of $w(x,t)$ and $\psi(x,t)$, respectively, $i=\sqrt{-1}$, ω_n is the natural frequency, ξ is the non-dimensional length and L the length of the beam.

Substituting Eq. (3) into the Eq. (1) and Eq. (2) and omitting the common factor $e^{i\omega_n t}$, one has

$$\frac{EI}{L^4}\left(1+\frac{G_p-P}{\kappa GA}\right)\frac{d^4W(\xi)}{d\xi^4}-\left(\rho A+\frac{\rho I k_w}{\kappa GA}\right)\omega_n^2 W(\xi)-\frac{1}{L^2}\left(\frac{EIk_w}{\kappa GA}+G_p-P\right)\frac{d^2W(\xi)}{d\xi^2}+$$
$$\frac{\rho I}{L^2}\left(1+\frac{E}{\kappa G}+\frac{G_p-P}{\kappa GA}\right)\omega_n^2\frac{d^2W(\xi)}{d\xi^2}+\frac{\rho^2 I}{\kappa G}\omega_n^4 W(\xi)+k_w W(\xi)=0, \tag{4}$$

$$\frac{EI}{L^4}\left(1+\frac{G_p-P}{\kappa GA}\right)\frac{d^4\Psi(\xi)}{d\xi^4}-\left(\rho A+\frac{\rho I k_w}{\kappa GA}\right)\omega_n^2\Psi(\xi)-\frac{1}{L^2}\left(\frac{EIk_w}{\kappa GA}+G_p-P\right)\frac{d^2\Psi(\xi)}{d\xi^2}+$$
$$\frac{\rho I}{L^2}\left(1+\frac{E}{\kappa G}+\frac{G_p-P}{\kappa GA}\right)\omega_n^2\frac{d^2\Psi(\xi)}{d\xi^2}+\frac{\rho^2 I}{\kappa G}\omega_n^4\Psi(\xi)+k_w\Psi(\xi)=0. \tag{5}$$

Inserting the following parameters

$$b^2=\frac{\rho AL^4}{EI}\omega_n^2,\ e^2=\frac{k_w L^4}{EI},\ n^2=\frac{PL^2}{EI},\ p^2=\frac{G_p L^2}{EI},\ r^2=\frac{I}{AL^2} \text{ and } s^2=\frac{EI}{\kappa GAL^2} \tag{6}$$

where b is the frequency parameter, e is the linear elastic springs parameter, n is the axial load parameter, p is the shear layer parameter, r is the rotational inertia parameter and s is the shear parameter. So one can write

$$W''''+\eta_1 W''+\eta_2 W=0 \qquad \text{and} \qquad \Psi''''+\eta_1\Psi''+\eta_2\Psi=0, \tag{7}$$

where

$$\eta_1=\frac{b^2(r^2+s^2)+b^2s^2r^2(p^2-n^2)-e^2s^2-p^2+n^2}{1+s^2(p^2-n^2)} \quad \text{and} \quad \eta_2=\frac{b^4s^2r^2-b^2(1+e^2s^2r^2)+e^2}{1+s^2(p^2-n^2)}. \tag{8}$$

Assuming the solution of the type $W(\xi)=Ce^{z\xi}$ and $\Psi(\xi)=\overline{C}e^{z\xi}$ and replacing in the Eqs. (7), one can obtain the following characteristic equation

$$z^4+\eta_1 z^2+\eta_2=0 \tag{9}$$

with $\eta_1^2+4\eta_2>0$. So the solutions can be written as

$$W(\xi)=C_1 cosh(s_1\xi)+C_2 sinh(s_2\xi)+C_3 cos(s_3\xi)+C_4 sin(s_4\xi), \tag{10}$$

$$\Psi(\xi)=\overline{C_1}cosh(s_1\xi)+\overline{C_2}sinh(s_2\xi)+\overline{C_3}cos(s_3\xi)+\overline{C_4}sin(s_4\xi). \tag{11}$$

In the Eq. (10) and Eq. (11) C_i and $\overline{C}_i$ are dependents constants and z_i are the roots of the Eq. (9).

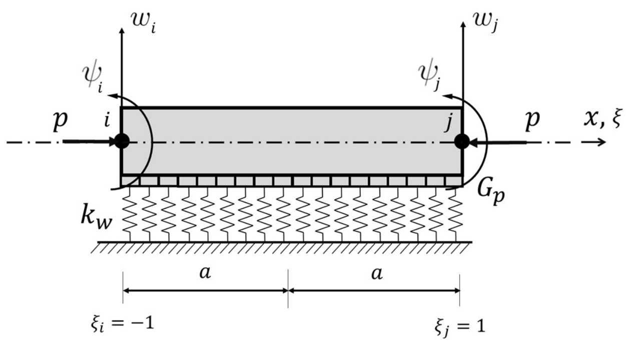

Figure 2. Beam element

3. FINITE ELEMENT METHOD

Consider a uniform axial-loaded Timoshenko beam element on Pasternak Foundation with length $2a$ as showed in Fig. 2. The beam element consists of two nodes and each node has two degrees of freedom: w_i, ψ_i, w_j and ψ_j.

Applying the Rayleigh-Ritz method using polynomial functions approximations and applying the boundary conditions on the points $\xi_i = -1$ and $\xi_j = 1$, the displacement and slope form matrices may be written as:

$$w = [\mathbf{N}(\xi)]_w \{\mathbf{v}\}_e \; and \; \psi = [\mathbf{N}(\xi)]_\psi \{\mathbf{v}\}_e \tag{12}$$

where $[\mathbf{N}(\xi)]_w$ and $[\mathbf{N}(\xi)]_\psi$ are the shape functions for displacement and slope, respectively, and $\{\mathbf{v}\}_e$ is the vector of nodal coordinates. The subscript e represents expressions for one single element. Therefore, the shape functions and the vector of nodal coordinates in Eq. (12), are expressed as:

$$[\mathbf{N}(\xi)]_w^T = \frac{1}{4(1+3\beta)} \begin{bmatrix} 2-3(1+2\beta)\xi + 6\beta + \xi^3 \\ a(1-\xi-(1+3\beta)\xi^2 + 3\beta + \xi^3) \\ 2+3(1+2\beta)\xi + 6\beta - \xi^3 \\ a(-1-\xi+(1+3\beta)\xi^2 - 3\beta + \xi^3) \end{bmatrix}, \tag{13}$$

$$[\mathbf{N}(\xi)]_\psi^T = \frac{1}{4(1+3\beta)} \begin{bmatrix} (-3+3\xi^2)/a \\ -1-(2+6\beta)\xi + 6\beta + 3\xi^3 \\ (3-3\xi^2)/a \\ -1+(2+6\beta)\xi + 6\beta + 3\xi^3 \end{bmatrix} \text{ and } \{\mathbf{v}\}_e^T = \; [w_i \quad \psi_i \quad w_j \quad \psi_j] \tag{14}$$

where $\beta = EI/(\kappa GAa^2)$.

Following Abohadima, the implementation of the axial load is analyzed using three approaches. In the first approach the angle α, measured between the cross-section and the axial load, is equal to zero, in the second approach it is equal to bending angle ψ and, lastly, in the third approach the angle is equal to shear angle plus the bending angle, that is:

$$\alpha_1 = 0, \; \alpha_2 = \psi, \text{ and } \alpha_3 = \frac{\partial w}{\partial x}. \tag{15}$$

Thus, considering the foundation, the axial load and the displacement and slope form matrices in the Eq. (12), the potential energy for an element $2a$ is given by:

$$\mathbf{U_e} = \frac{1}{2}\{\mathbf{v}\}_e^T \Bigg[\frac{EI}{a}\int_{-1}^{1}[\mathbf{N}'(\xi)]_\psi^T[\mathbf{N}'(\xi)]_\psi d\xi + \frac{\kappa GA}{a}\int_{-1}^{1}\Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big]^T\Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big]d\xi +$$
$$k_w a\int_{-1}^{1}[\mathbf{N}(\xi)]_w^T[\mathbf{N}(\xi)]_w d\xi + \frac{G_p}{a}\int_{-1}^{1}[\mathbf{N}'(\xi)]_w^T[\mathbf{N}'(\xi)]_w d\xi + \frac{P}{a}\int_{-1}^{1}[\mathbf{\Gamma}(\xi)]_\gamma^T[\mathbf{\Gamma}(\xi)]_\gamma d\xi\Bigg]\{\mathbf{v}\}_e, \tag{16}$$

where $[\mathbf{\Gamma}(\xi)]_\gamma = 0$ in the first approach, $[\mathbf{\Gamma}(\xi)]_\gamma = [\mathbf{N}(\xi)]_\psi$ in the second approach, $[\mathbf{\Gamma}(\xi)]_\gamma = [\mathbf{N}'(\xi)]_w$ in the third approach and $[\mathbf{N}'(\xi)] = \partial[\mathbf{N}(\xi)]/\partial\xi$. And the kinetic energy is given by:

$$\mathbf{T_e} = \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T\left[\rho Aa\int_{-1}^{1}[\mathbf{N}(\xi)]_w^T[\mathbf{N}(\xi)]_w d\xi + \rho Ia\int_{-1}^{1}[\mathbf{N}(\xi)]_\psi^T[\mathbf{N}(\xi)]_\psi d\xi\right]\{\dot{\mathbf{v}}\}_e \tag{17}$$

where $\{\dot{\mathbf{v}}\} = \partial\{\mathbf{v}\}/\partial t$. Therefore, the elementary stiffness and mass matrices are given by:

$$[\mathbf{k_e}] = \frac{EI}{a}\int_{-1}^{1}[\mathbf{N}'(\xi)]_\psi^T[\mathbf{N}'(\xi)]_\psi d\xi + \frac{\kappa GA}{a}\int_{-1}^{1}\Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big]^T\Big[[\mathbf{N}'(\xi)]_w - a[\mathbf{N}(\xi)]_\psi\Big]d\xi +$$

$$k_w a \int_{-1}^{1} [\mathbf{N}(\xi)]_w^T [\mathbf{N}(\xi)]_w d\xi + \frac{G_p}{a} \int_{-1}^{1} [\mathbf{N}'(\xi)]_w^T [\mathbf{N}'(\xi)]_w d\xi + \frac{P}{a} \int_{-1}^{1} [\mathbf{N}'(\xi)]_w^T [\mathbf{N}'(\xi)]_w d\xi, \quad (18)$$

$$[\mathbf{m_e}] = \rho A a \int_{-1}^{1} [\mathbf{N}(\xi)]_w^T [\mathbf{N}(\xi)]_w d\xi + \rho I a \int_{-1}^{1} [\mathbf{N}(\xi)]_\psi^T [\mathbf{N}(\xi)]_\psi d\xi. \quad (19)$$

4. NUMERICAL RESULTS

As the first comparison, let one consider the beams with constant cross-section, subjected to axial forces as studied by Abohadima *et al.* (2015) by means of an analytical method. Taking Poisson coefficient equal to $1/4$, $E/G = 2.5$, the shear factor given by $\kappa = 2/3$, $\rho = 2500kg/m^3$, $I = 0,05^2 m^4$ and $A = 1m^2$. The natural frequency parameters b_i of the lower three modes for Timoshenko beams are obtained by FEM (discretization with 30 and 50 elements) using the eigenvalue problem and they are compared with the exact solution gives by Abohadima.

Table 1. First three frequencies parameters for a hinged-hinged beam.

Parameters			b_1			b_2			b_3		
n	e	p	30 ele.	50 ele.	Exact$^{(1)}$	30 ele.	50 ele.	Exact$^{(1)}$	30 ele.	50 ele.	Exact$^{(1)}$
0	0	0	2.866	2.866	2.866	4.925	4.923	4.922	6.454	6.449	6.446
0.6	0	0	1.862	1.862	1.863	4.387	4.384	4.384	5.932	5.926	5.923
0.6	$0.6\pi^4$	0	2.866	2.866	2.866	4.540	4.538	4.538	5.997	5.991	5.988
0.6	$0.6\pi^4$	π^2	3.555	3.555	3.555	5.296	5.294	5.294	6.785	6.779	6.777

$^{(1)}$ (Abohadima *et al.*, 2015)

To analyze the influence due to the axial load and due to the foundation parameters in the frequency parameters, Tab. 1 presents the comparison between exact and FEM solutions for a hinged-hinged beam.

The results show that the presence of axial load decreases the frequency parameters, this is because the transverse component of the axial load causes an increase in shear. By comparing the frequency parameter of a beam on a foundation subjected to axial load with the solution of a beam in the same conditions without foundation, the addition of the first foundation parameter causes a significant increase in the frequency parameter in the first mode, but becomes less significant in each higher mode. In other hand, the second foundation parameter presents a remarkable increase for all modes of vibration showed, being superior to those presented by the first foundation parameter. In addition, the variations in the frequency parameter decreases as the mode number increases.

For higher modes, as expected, the difference between FEM and analytical solutions decreases when the number of elements is increased. The biggest error noticed in this comparison was 0,152% for the third frequency with axial and without foundation and using 30 elements.

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