SECOND SPECTRUM OF TIMOSHENKO BEAM ON PASTERNAK FOUNDATION

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Wallison Kennedy da Silva Bezerra
Universidade Federal do Piauí

Lucas S. Soares
Universidade Federal do Piauí

Simone dos Santos
Universidade Federal do Piauí

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Wallison Kennedy da Silva Bezerra

Lucas Silva Soares

Simone dos Santos Hoefel

wallisonkennedy1@hotmail.com

soares.l.s.2373@gmail.com

simone.santos@ufpi.edu.br

Laboratório de Mecânica Computacional

Universidade Federal do Piauí, Campus Universitário Ministro Petrônio Portela - Bairro Ininga, CEP: 64049-550, Teresina, Piauí, Brasil.

Abstract. *In many soil-structure interaction problems, the soil medium model used is the Winkler foundation for mathematical simplicity. However, this foundation model cannot represent the behavior of foundation materials for all engineering applications. The Pasternak model can accomplish a more realistic and generalized description of the soil behavior. For structures with a large aspect ratio of height and length, Timoshenko beam theory is used, instead of Euler-Bernoulli theory, since it takes both shear and rotary inertia into account. This paper investigates the effects of the Pasternak foundation on the dynamic response of the Timoshenko beam. A finite element is developed using cubic and quadratic polynomials, which are made interdependent by requiring them to satisfy the static homogeneous differential equations associated with Timoshenko beam theory. The influence of the foundation on the second spectrum is concerned. The results showed that the presence of the foundation anticipates the second spectrum and does not have a significant influence in its frequencies. Also, the presence of the foundation permits a distinction between the two spectra for boundary conditions that do not factorize.*

Keywords: *Finite Element Method, Timoshenko Beam Theory, Pasternak Foundation, Second Spectrum, Free Vibration Analysis*

1 INTRODUCTION

The issue of a beam on elastic foundation gained the interest of researchers and engineers due to its importance in structural dynamics and soil media behavior description, as it can simulate geotechnical structures and transportation systems. The dynamic characteristics of these structures constitute a fundamental procedure to determine the presence of undesired vibrations which can affect the passenger comfort or adjacent structures.

Due to the very complex characteristics of soil-media, various researchers made efforts in an attempt to propose a simple mathematical and physically consistent model to represent the soil-structure interaction (Dutta and Roy, 2002). To obtain a meaningful and reliable information for the response of the soil media, the models for practical problems takes into account only some specifics aspects of its behavior (Selvadurai, 1979).

The simplest foundation model is that proposed by Winkler (1867), in which it the foundation is represented as a series of mutually independent, closely spaced, discrete, linearly elastic springs without coupling effects between each other (Dutta and Roy, 2002). However, this model is a crude approximation of the correct mechanical behavior of the soil media, as it disregards the effect of continuity and cohesion of the ground. Hence, this model cannot represent the behavior of foundation materials for all engineering applications, giving a reliable response for soils with lower cohesion characteristics. Thus, two-parameter proposed by Pasternak (1954) improves the Winkler model by adding a shear layer, a structural element that deforms only due to transverse shear, to guarantee the interaction among the separated springs (Avramidis and Morfidis, 2006). This approach provides more reliable information of stresses and deformations of the soil media.

As far as the behavior of the beam is concerned, the most applications bases in the classical Euler-Bernoulli theory in which predicts that straight lines and normal planes to neutral axis remains after deformation (Avramidis and Morfidis, 2006). This model gives a good accuracy for lower frequencies of a thin beam, however, in higher modes, the frequencies become inaccurate.

To overcome this limitation, the Timoshenko beam theory, which takes into account the contribution of shear deformation and rotary inertia correction, gives a reliable information for thicker beams and higher natural frequencies. The main consequence of this theory is the existence of two frequency spectra (Manevich, 2015). This phenomenon that occurs mainly for some boundary conditions intrigued researchers over the years.

Several papers concerned investigations this field. Traill-Nash and Collar (1953) first noted the presence of the second spectrum for higher modes. Downs (1976) concerned the subject and showed the difference of phase between the bending and shear deformation. Abba and Thomas (1978) used a finite element model for the stability analysis of a Timoshenko beam resting on an elastic foundation and its effect in natural frequencies, in which the effect of foundation in the second spectrum was only suggested. Bhashyam (1981) showed that the second spectrum exists for boundary conditions other than hinged-hinged.

Stephen (2006a, 2006b) presented a wave propagation analysis and compared with the exact solution from plane stress elastodynamic theory, experimental results, and a finite element solution. These works showed an inconsistent behavior of the second spectrum and concluded that it should be disregarded. Manevich (2015) analyzed the free transverse waves in Timoshenko beam resting on Winkler foundation and gave and particular attention to clearing up the physical sense of the second spectrum of Timoshenko beam under this type of interaction. Azevedo et al. (2016a, 2016b) discussed the influence of rotational inertia and shear deformation in the second spectrum and its characteristics for different boundary conditions.

The primary objective of this paper is to comprehend the effect of elastic foundation in the occurrence of the second spectrum. A finite element with two nodes and two degrees of freedom per nodes is developed with cubic and quadratic for the transverse displacement and the slope due to bending, respectively. The polynomials are made interdependent by requiring them to satisfy both of the static homogeneous differential equations associated with TBT. Numerical examples present the influence of the Pasternak foundation on the dynamic response of the Timoshenko beam for various boundary conditions. The influence of the foundation on the second spectrum is concerned.

2 TIMOSHENKO BEAM MODEL

Timoshenko theory is a major improvement for non-slender beams and for high-frequency responses where shear or rotary effects are not negligible, the governing decoupled differential equations for transverse vibrations are (Soares and Hoefel, 2015):

$$\frac{\partial^4 V(\xi)}{\partial \xi^4} + b^2 s^2 \frac{\partial^2 V(\xi)}{\partial \xi^2} + b^2 r^2 \frac{\partial^2 V(\xi)}{\partial \xi^2} + b^4 r^2 s^2 V(\xi) - b^2 V(\xi) = 0, \tag{1}$$

$$\frac{\partial^4 \Psi(\xi)}{\partial \xi^4} + b^2 s^2 \frac{\partial^2 \Psi(\xi)}{\partial \xi^2} + b^2 r^2 \frac{\partial^2 \Psi(\xi)}{\partial \xi^2} + b^4 r^2 s^2 \Psi(\xi) - b^2 \Psi(\xi) = 0, \tag{2}$$

where $\xi = x/L,\ b^2 = \frac{\rho A L^4}{EI}\omega^2$. ξ is the non-dimensional length of the beam, and $V(x)$ and $\Psi(x)$ are the normal functions of displacement and slope due to bending, respectively. r and s are coefficients related with the effect of rotatory inertia and shear deformation given by:

$$r^2 = \frac{I}{AL^2} \qquad \text{and} \qquad s^2 = \frac{EI}{\kappa AGL^2}, \tag{3}$$

We must consider two cases when obtaining Timoshenko beam model spatial solution. In the first case, assume:

$$\sqrt{(r^2 - s^2)^2 + 4/b^2} > (r^2 + s^2) \quad \text{which leads to} \quad b < \frac{1}{rs}, \tag{4}$$

while in the second

$$\sqrt{(r^2 - s^2)^2 + 4/b^2} < (r^2 + s^2) \quad \text{which leads to} \quad b > \frac{1}{rs}. \tag{5}$$

Equations 4 and 5 presents the conditions to distinguish two behaviors of the Timoshenko beam (Azevedo et al., 2016a, 2016b). The critical value is given by $b_{crit} = 1/(rs)$. Substituting this values in the relation presented for the natural frequency, we have the critical frequency expressed as:

$$\omega_{crit} = \sqrt{\frac{\kappa GA}{\rho I}}. \tag{6}$$

When $b < b_{crit}$, the solution of Eqs. 1 and 2 can be expressed respectively in trigonometric and hyperbolic functions:

$$V(\xi) = C_1 cosh(\alpha_1 \xi) + C_2 sinh(\alpha_1 \xi) + C_3 cos(\beta\xi) + C_4 sin(\beta\xi), \tag{7}$$
$$\Psi(\xi) = C'_1 sinh(\alpha_1 \xi) + C'_2 cosh(\alpha_1 \xi) + C'_3 sin(\beta\xi) + C'_4 cos(\beta\xi), \tag{8}$$

with

$$\alpha_1 = \frac{b}{\sqrt{2}} \left[-(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}} \right]^{1/2}, \tag{9}$$

$$\beta = \frac{b}{\sqrt{2}} \left[(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}} \right]^{1/2} \tag{10}$$

and C and C' are constants.

Equations 7 and 8 have two eigenvalues, α_1 and β, that are related with trigonometric and hyperbolic sines and cosines, respectively. When $b > b_{crit}$ the solution $V(\xi)$ and $\Psi(\xi)$ can be expressed only in trigonometric functions:

$$V(\xi) = \overline{C}_1 cos(\alpha_2 \xi) + \overline{C}_2 sin(\alpha_2 \xi) + \overline{C}_3 cos(\beta\xi) + \overline{C}_4 sin(\beta\xi), \tag{11}$$
$$\Psi(\xi) = \overline{C}'_1 sin(\alpha_2 \xi) + \overline{C}'_2 cos(\alpha_2 \xi) + \overline{C}'_3 sin(\beta\xi) + \overline{C}'_4 cos(\beta\xi), \tag{12}$$

with

$$\alpha_2 = \frac{b}{\sqrt{2}} \left[(r^2+s^2) - \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}} \right]^{1/2}, \tag{13}$$

$$\beta = \frac{b}{\sqrt{2}} \left[(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}} \right]^{1/2} \tag{14}$$

and $\bar{C}$ and $\bar{C}'$ are constants. The relations between the constants in Eqs. 7 and 8 or Eqs. 11 and 12 can be found in Huang (1961).

The natural frequency ω is written in terms of two eigenvalues (α_1 and β or α_2 and β) as follows (Huang, 1961):

$$\omega_i = \frac{\sqrt{\beta^2 - \alpha_1^2}}{\sqrt{r^2+s^2}} \left(\frac{EI}{\rho A L^4} \right)^{1/2} \quad \text{with} \quad i = 1, 2, ..., n. \quad \text{when} \quad b < b_{crit}, \tag{15}$$

$$\omega_i = \frac{\sqrt{\beta^2 + \alpha_2^2}}{\sqrt{r^2+s^2}} \left(\frac{EI}{\rho A L^4} \right)^{1/2} \quad \text{with} \quad i = 1, 2, ..., n. \quad \text{when} \quad b > b_{crit}. \tag{16}$$

Table 1 presents frequency equations obtained considering $b < b_{crit}$ and $b > b_{crit}$ for clamped-clamped (c-c) and hinged-hinged (h-h) Timoshenko beam.

Table 1: Frequency equations of Timoshenko model

	frequency equation ($b < b_{crit}$)
c-c	$2 - 2cosh(\alpha_1)cos(\beta) + \frac{b(b^2s^2(r^2-s^2)^2 + (3s^2-r^2))}{\sqrt{1-b^2r^2s^2}} sinh(\alpha_1)sin(\beta) = 0$
h-h	$sin(\beta)sinh(\alpha_1) = 0$
	frequency equation ($b > b_{crit}$)
c-c	$2 - 2cos(\alpha_2)cos(\beta) + \frac{b(b^2s^2(r^2-s^2)^2 + (3s^2-r^2))}{\sqrt{b^2r^2s^2-1}} sin(\alpha_2)sin(\beta) = 0$
h-h	$sin(\alpha_2)sin(\beta) = 0$

3 PASTERNAK FOUNDATION MODEL

The equation of motion can be derived in the general case in which the Timoshenko beam lays on a Pasternak foundation. Fig. 1 shows a scheme of uniform Timoshenko beam on a Pasternak foundation. The uncoupled differential equations can be expressed in an analytical form as (De Rosa, 1995; Wang and Stephens, 1977)

$$\frac{d^4V(\xi)}{d\xi^4} + \gamma\frac{d^2V(\xi)}{d\xi^2} + \zeta V(\xi) = 0, \tag{17}$$

$$\frac{d^4\Psi(\xi)}{d\xi^4} + \gamma\frac{d^2\Psi(\xi)}{d\xi^2} + \zeta \Psi(\xi) = 0, \tag{18}$$

where

$$\gamma = \frac{b^2\left(r^2+s^2\right) - s^2e^2 + p^2\left(b^2r^2s^2-1\right)}{1+s^2p^2}, \quad \zeta = \frac{\left(b^2-e^2\right)\left(b^2r^2s^2-1\right)}{1+s^2p^2}$$

and e and p are the coefficients related with foundation stiffness and foundation shear, respectively, given by:

$$e^2 = \frac{k_fL^4}{EI}, \; p^2 = \frac{G_pL^2}{EI}. \tag{19}$$

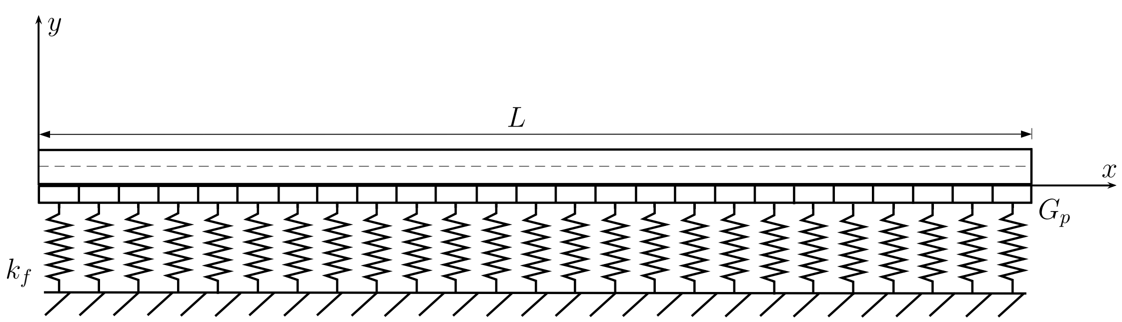

Figure 1: A beam on a Pasternak foundation.

To solve the O.D.E. of Eqs. 17 and 18, two conditions must be considered. In the first case, assume:

$$\zeta < 0, \text{ which leads to: } b < \frac{1}{rs} \text{ and } b > e \text{ or } b > \frac{1}{rs} \text{ and } b < e. \tag{20}$$

This condition results in the solutions to be expressed in trigonometric and hyperbolic functions:

$$V(\xi) = C_1\, cosh(\alpha_1 \xi) + C_2\, sinh(\alpha_1 \xi) + C_3\, cos(\beta \xi) + C_4\, sin(\beta \xi), \tag{21}$$

$$\Psi(\xi) = C'_1\, sinh(\alpha_1 \xi) + C'_2\, cosh(\alpha_1 \xi) + C'_3\, sin(\beta \xi) + C'_4\, cos(\beta \xi), \tag{22}$$

where:

$$\alpha_1 = \frac{\sqrt{2}}{2}\sqrt{-\gamma + \sqrt{\gamma^2 - 4\zeta}} \tag{23}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\gamma + \sqrt{\gamma^2 - 4\zeta}}, \tag{24}$$

and C and C' are constants.

The second case gives:

$$\zeta > 0, \text{ which leads to: } b > \frac{1}{rs} \text{ and } b > e \text{ or } b < \frac{1}{rs} \text{ and } b < e. \tag{25}$$

As a result, the solution is expressed only in trigonometric functions:

$$V(\xi) = \bar{C}_1\, cos(\alpha_2 \xi) + \bar{C}_2\, sin(\alpha_2 \xi) + \bar{C}_3\, cos(\beta \xi) + \bar{C}_4\, sin(\beta \xi), \tag{26}$$

$$\Psi(\xi) = \bar{C}'_1\, sin(\alpha_2 \xi) + \bar{C}'_2\, cos(\alpha_2 \xi) + \bar{C}'_3\, sin(\beta \xi) + \bar{C}'_4\, cos(\beta \xi), \tag{27}$$

where:

$$\alpha_2 = \frac{\sqrt{2}}{2}\sqrt{\gamma - \sqrt{\gamma^2 - 4\zeta}}, \tag{28}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\gamma + \sqrt{\gamma^2 - 4\zeta}}, \tag{29}$$

and $\bar{C}$ and $\bar{C}'$ are constants.

Equations 17 and 18 shows that the beam-foundation theory represents a generalization of the beam theory. Disregarding the parameters e and p, the solution goes back to the solution of a beam without foundation. Also, the Pasternak foundation theory is a higher generalization as it includes the solution for Winkler when $p = 0$.

4 FINITE ELEMENT FORMULATION

Consider a uniform Timoshenko beam element on Pasternak Foundation as shown in Fig. 2. The beam element consists of two nodes and each node has two degrees of freedom: V, the total deflection, and Ψ, the slope due to bending.

Solving the homogeneous form of Timoshenko beam static equations, one can obtain a cubic and quadratic displacement functions as follows (Yokoyama, 1987):

$$V_i(\xi) = \sum_{i=0}^{3} \lambda_i \xi^i \quad \text{and} \quad \Psi_i(\xi) = \sum_{i=0}^{2} \overline{\lambda}_i \xi^i. \tag{30}$$

where λ_i and $\overline{\lambda}_i$ are constants.

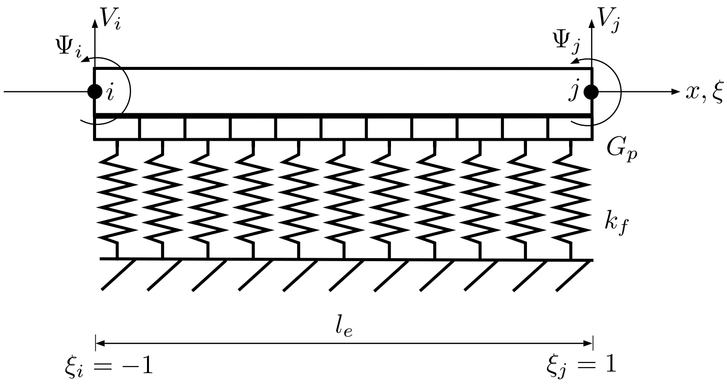

Figure 2: Beam on Pasternak foundation element

Using the non-dimension coordinate, ξ, and element length, l_e. 2, the matrix form of the displacement V and total slope Ψ can be written as:

$$V = [\,\mathbf{N}(\xi)\,]\{\,\mathbf{v}\,\}_e \quad \text{and} \quad \Psi = [\,\overline{\mathbf{N}}(\xi)\,]\{\,\mathbf{v}\,\}_e, \tag{31}$$

where $[\,\mathbf{N}(\xi)\,]$ and $[\,\overline{\mathbf{N}}(\xi)\,]$ are the shape functions and $\{\,\mathbf{v}\,\}_e$ is the vector of nodal coordinates. The subscript e represents expressions for a single element.

Therefore, the shape functions in Eq. 31 can be expressed as:

$$\mathbf{N_i}(\xi) = \frac{1}{4(1+3\beta)} \begin{bmatrix} 2(3\beta+1) - 3(\beta+1)\xi + \xi^3 \\ (l_e/2)\,[3\beta+1-\xi-(3\beta+1)\xi^2+\xi^3] \\ 2(3\beta+1) + 3(2\beta+1)\xi - \xi^3 \\ (l_e/2)\,[-3\beta-1-\xi+(3\beta+1)\xi^2+\xi^3] \end{bmatrix}^T, \tag{32}$$

and

$$\overline{\mathbf{N}}_i(\xi) = \frac{1}{4(1+3\beta)} \begin{bmatrix} (l_e/2)\,(3\xi^2-3) \\ -1-2(3\beta+1)\xi+6\beta+3\xi^2 \\ (l_e/2)\,(3-3\xi^2) \\ -1+2(3\beta+1)\xi+6\beta+3\xi^2 \end{bmatrix}^T, \tag{33}$$

where $\beta = 4EI/\kappa G A {l_e}^2$.

Thus, considering the foundation and the beam, the potential and kinetic energy for an element length l_e are given by:

$$\begin{aligned} \boldsymbol{U_e} &= \frac{1}{2}\frac{2EI}{l_e}\int_{-1}^{1}\left(\frac{\partial \Psi}{\partial \xi}\right)^2 d\xi + \frac{1}{2}\frac{2\kappa G A}{l_e}\int_{-1}^{1}\left(\frac{2}{l_e}\frac{\partial V}{\partial \xi} - \Psi\right)^2 d\xi \; + \\ & \quad \frac{1}{2}\frac{k_f l_e}{2}\int_{-1}^{1}(V)^2\, d\xi + \frac{1}{2}\frac{2G_p}{l_e}\int_{-1}^{1}\left(\frac{\partial V}{\partial \xi}\right)^2 d\xi \end{aligned} \tag{34}$$

$$\boldsymbol{T_e} = \frac{1}{2}\frac{\rho A l_e}{2}\int_{-1}^{1}\left(\frac{\partial V}{\partial t}\right)^2 d\xi + \frac{1}{2}\frac{\rho I l_e}{2}\int_{-1}^{1}\left(\frac{\partial \Psi}{\partial t}\right)^2 d\xi. \tag{35}$$

Substituting the displacement expression, Eq. 31, into the potential energy, Eq. 35, gives:

$$\begin{aligned} \boldsymbol{U_e} \;=\; & \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{2EI}{l_e}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi\right]\{\mathbf{v}\}_e + \\ & \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{2\kappa GA}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)' - \frac{l_e}{2}\overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)' - \frac{l_e}{2}\overline{\mathbf{N}}(\xi)]d\xi\right]\{\mathbf{v}\}_e + \\ & \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{k_f l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\,d\xi + \frac{2G_p}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']\,d\xi\right]\{\mathbf{v}\}_e, \end{aligned} \tag{36}$$

where $[\mathbf{N}(\xi)'] = [\partial\mathbf{N}(\xi)/\partial\xi]$.

Substituting the displacement expression, Eq. 31, into the kinetic energy, Eq. 35, gives:

$$\boldsymbol{T_e} = \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T\left[\frac{\rho A l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi + \frac{\rho A l_e}{2}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]d\xi\right]\{\dot{\mathbf{v}}\}_e, \tag{37}$$

5 NUMERICAL RESULTS

To show the effects of the foundation parameters on the natural frequencies of a beam on elastic foundation, some numerical examples are presented. The same parameter values are used for all examples, except when specified. Consider a beam of uniform cross-section, such that $E = 210\,GPa$, $G = 80.8\,GPa$, $\rho = 7850\,kg/m^3$, and $L = 0.5\,m$. Méndez-Sánchez et al. (2005) presented, from comparison with experimental results, that the value for the shear correction factor $\kappa = 5(1+\nu)/(6+5\nu)$ presents the best accuracy when considering the one-parameter approach. Therefore, adopting this shear correction factor, $\nu = E/(2G) - 1 = 0.30$ leads to $\kappa = 0.867$.

Table 2: Comparison table for the frequency parameters of analytic and FEM analyses.

Mode Number	Without Foundation	Winkler				Pasternak			
		Analytic	FEM-10e	FEM-30e	FEM-70e	Analytic	FEM-10e	FEM-30e	FEM-70e
1	9.5752	10.786	10.787	10.786	10.786	18.962	18.963	18.962	18.962
2	35.410	35.746	35.834	35.756	35.748	47.134	47.206	47.142	47.136
3	71.842	72.003	72.726	72.082	72.018	85.151	85.807	85.222	85.164
4	114.26	114.36	117.22	114.67	114.41	129.18	131.91	129.48	129.24
5	159.85	159.92	167.67	160.78	160.08	176.56	184.11	177.39	176.71

To analyze the influence of the presence of a foundation in the frequency parameters, Table 2 presents the comparison between analytic and FEM solutions with $r = 0.04$ for a hinged-hinged beam. The second column presents the frequency parameters for a beam without foundation ($e = 0$, $p = 0$), third to sixth, for Winkler foundation ($e = 5$, $p = 0$), and seventh to tenth, for Pasternak foundation ($e = 5, p = 5$).

The results showed that the presence of a foundation increase the frequency parameter and Pasternak presents the higher increase. Also, the Winkler increase reduces drastically as the mode number rise. This reduction can also be observed in the frequency parameters of Pasternak foundation. However, the decrease is not as high as presented by Winkler.

For higher modes, the difference between the FEM and analytic solutions decreases when the number of elements is increased. Therefore, FEM formulation presents a high accuracy.

The subsections present numerical examples for hinged-hinged and clamped-clamped beams to analyze the effect of the Pasternak foundation in the different spectra of a Timoshenko beam. The boundary conditions are defined in Table 3.

Table 3: Boundary Conditions

Boundary Condition	Deflection	Slope	Moment	Shear Force
Hinged	$V(\xi) = 0$	-	$\dfrac{\partial \Psi(\xi)}{\partial \xi} = 0$	-
Clamped	$V(\xi) = 0$	$\Psi(\xi) = 0$	-	-

5.1 Hinged-hinged beam

Tables 4 and 5 shows the natural frequencies for the first spectrum (TBT1) and the second spectrum (TBT2), respectively, for a hinged-hinged beam without foundation ($e = 0$, $p = 0$, and $\kappa = 5/6$) and a beam on a Pasternak foundation ($e = 5$, $p = 5$, and $\kappa = 0.867$), both with $r = 0.0722$. The example provides 10 natural frequencies: the first five frequencies for TBT1, the first four frequencies for TBT2, and the shear mode frequency. The results presented for a beam without foundation corresponds to the presented by Azevedo et al. (2016b) as they noticed that Levinson and Cooke (1982) appear to present a typographic error for the shear mode value.

Table 4: Natural frequencies for the first spectrum of the hinged - hinged Timoshenko beam on Pasternak foundation (rad/s)

Without Foundation				Pasternak			
Mode	TBT1	FEM - 70e		Mode	TBT1	FEM - 70e	
Number	Frequency	Frequency	Error (%)	Number	Frequency	Frequency	Error (%)
1	6712	6712.53	0.0078	1	13811.77	13811.82	0.0004
2	22136	22139.61	0.0163	2	31754.35	31756.56	0.0069
3	40701	40720.56	0.0481	3	52820.30	52835.52	0.0288
4	60170	60227.69	0.0959	4	75052.86	75105.04	0.0695
5	79806	79936.26	0.1632	Shear Mode	81164	81205.98	0.0517
Shear Mode	81164	81205.98	0.0517	5	97669.76	97796.62	0.1299

The results showed that the significant increase in the TBT1 frequencies anticipates the appearance of the TBT2. This behavior is due to the significant increase in the lower modes that makes the frequencies near to the shear mode surpass it. The increase in the TBT2 frequencies

Table 5: Natural frequencies for the second spectrum of the hinged - hinged Timoshenko beam on Pasternak foundation (rad/s)

Without Foundation				Pasternak			
Mode	TBT2	FEM - 70e		Mode	TBT2	FEM - 70e	
Number	Frequency	Frequency	Error (%)	Number	Frequency	Frequency	Error (%)
Shear Mode	81164	81205.98	0.0517	Shear Mode	81164	81205.98	0.0517
1	89077	89151.54	0.0837	1	89123.14	89186.38	0.0710
2	108044	108174.10	0.1204	2	108268.71	108395.04	0.1167
3	132212	132442.17	0.1741	3	132671.48	132909.75	0.1796
4	158992	159378.98	0.2434	4	159638.75	160048.73	0.2568

is minimal when compared with increases for TBT1. Also, the results are in good agreement with those obtained by Azevedo et al. (2016b), ratifying the reliability of the FEM solution.

This result demonstrates an adverse behavior of the influence of the foundation in the frequencies of a Timoshenko beam, as the first frequency of the TBT1 above the shear mode still has a significant influence of the foundation. This fact shows that the foundation does not present a significant influence only in TBT2 for frequencies above the critical, as was hinted by Abbas and Thomas (1978).

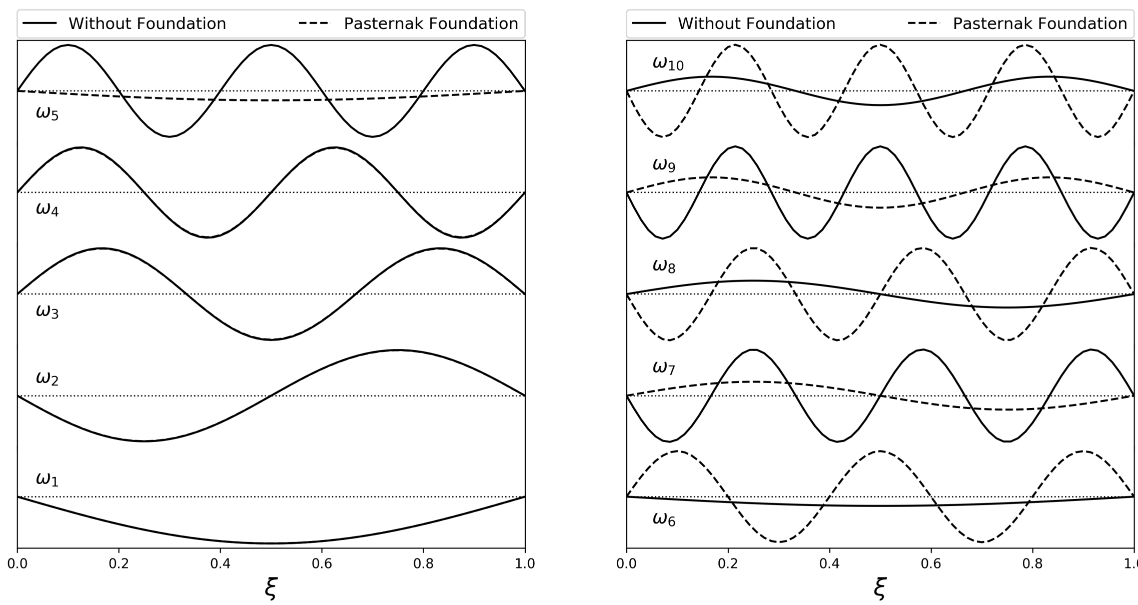

Figure 3: Comparison for the first 10 natural frequencies between a hinged-hinged beam without foundation and on Pasternak foundation ($e = 5$, $p = 5$) **using 70 elements**

The figure 3 shows the behavior of the modes shapes for a beam on Pasternak foundation. The presence of the Pasternak foundation appears to not influence in the mode shapes. The two distinct behavior below and above the critical frequency for the mode shapes still present. This distinction is due to the factorization of the frequency equation as shown in Table 1. Above the critical frequency, the mode presents two eigenvalues, one associated with $sin(\alpha_2) = 0$ and other with $sin(\beta) = 0$, which generates two distinct modes (Levinson and Cooke, 1982).

The difference between the amplitude of the modes from TBT1 and TBT2 is due to the difference in phase between the bending and shear deformation. The deformations due to shear and deflection are in the same phase for TBT1 and are summed to give the total amplitude. For TBT2, the shear and bending deformation are opposed with the amplitude equal to their difference (Downs, 1976).

5.2 Clamped-clamped beam

Table 6 shows the first 10 natural frequencies and the critical frequency for a clamped-clamped beam, in which $r = 0.0722$ and $\kappa = 0.867$, without foundation ($e = 0$, $p = 0$) and on Pasternak ($e = 5$, $p = 5$) using 70 elements.

Table 6: Natural frequencies for clamped-clamped Timoshenko beam on Pasternak foundation (rad/s)

Without Foundation		Pasternak Foundation	
Mode Number	Frequency	Mode Number	Frequency
1	12287.17	1	17524.31
2	26913.73	2	35532.31
3	43935.12	3	55404.86
4	61854.50	4	75867.73
5	80485.92	Critical Frequency	81163.51
Critical Frequency	81163.51	5	89591.95
6	88741.16	6	96786.51
7	100196.61	7	109680.43
8	107438.89	8	119348.05
9	120106.18	9	134117.56
10	130881.59	10	142587.62

The tables show that the foundation anticipates the critical frequency by increasing the natural frequencies, as seen in the hinged-hinged case. Also, for some frequencies above the critical, the increase is not notorious as showed for frequencies below the critical.

The figure 4 shows that the modes shapes for a beam on Pasternak foundation present a slightly different from the modes shapes without foundation. Also, the mode shapes still present the behavior observed by Smith (2008), in which the number of peaks in the mode shapes increases by a unit with the increase in mode number for frequencies lower than the critical, but for frequencies above, the number of peaks increases with each pair of modes.

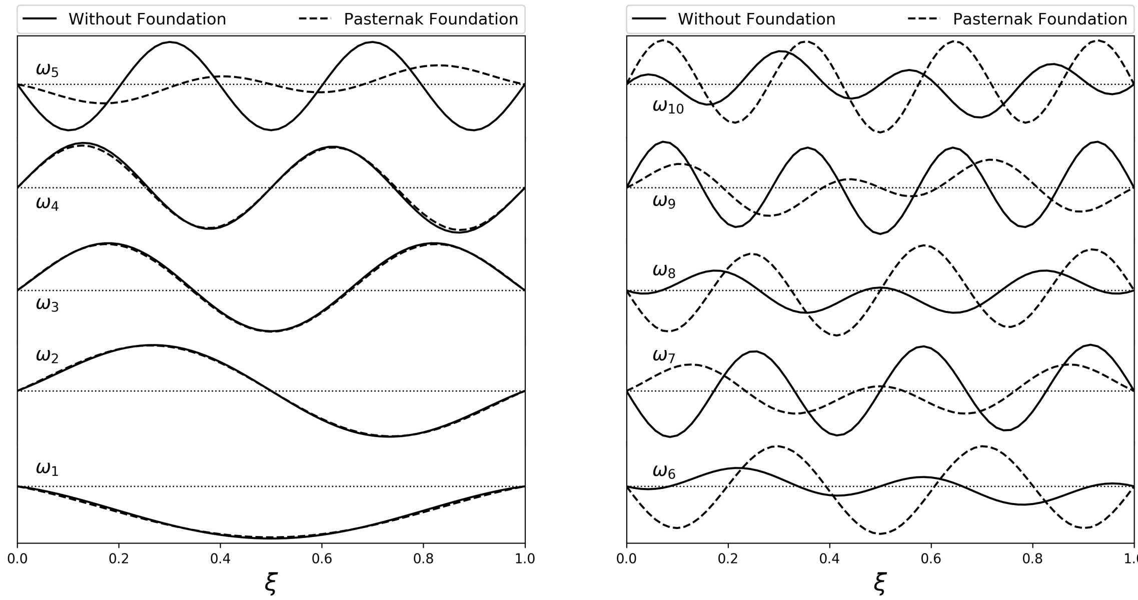

Figure 4: Comparison for the first 10 natural frequencies between a clamped-clamped beam without foundation and on Pasternak foundation ($e = 5$, $p = 5$) using 70 elements

For the clamped-clamped beam, the characteristic function does not factorizes, as can be seen in Table 1, thus, the mode shapes does not present two explicit distinguishable spectra. However, as the foundation does not present a significant influence in the TBT2 frequencies for the hinged-hinged case, the difference in the influence between frequencies allows inferring that the frequencies with a lower increase are from the TBT2 and those with a more substantial increase belongs to TBT1.

Therefore, even though the clamped-clamped boundary condition does not present the factorization that characterizes and distinguish the second spectrum, there is some difference between the behavior below and above the critical frequency and between the frequencies above the critical. These differences permit considering that the clamped-clamped beam presents the phenomena of the second spectra as shown by the hinged-hinged beam, even though it is not explicit.

6 CONCLUSIONS

In this paper, the full development and analysis of TBT for the transversely vibrating uniform beam were presented for classical boundary condition. A finite element is developed with cubic and quadratic polynomials for the total deflection and slope, respectively, where the polynomials are made interdependent by requiring them to satisfy both of the static homogeneous TBT differential equations. Numerical examples are presented for boundary conditions to study beam on Pasternak foundation behavior above critical frequency. The results showed that the presence of the foundation anticipates the second spectrum and does not have a significant influence in its frequencies. For a clamped-clamped beam, the foundation slightly affects the mode shapes. Also, the presence of the foundation permits a distinction between the two spectra for boundary conditions that do not factorize.

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