CONEM 2016 CONGRESSO NACIONAL DE ENGENHARIA MECÂNICA
21 - 25 AGOSTO DE 2016 FORTALEZA - CEARÁ
ABCM 40 anos Associação Brasileira de Engenharia e Ciências Mecânicas

ANALYSIS OF ROTATORY INERTIA AND SHEAR DEFORMATION ON TRANSVERSE VIBRATION OF BEAMS

Anderson Soares da Costa Azevêdo, anderson4r2@hotmail.com
Simone dos Santos Hoefel, simone.santos@ufpi.edu.br

Universidade Federal do Piauí, Campus Universitário Ministro Petrônio Portella, Teresina - PI, 64049-550

Abstract: *In the classical Bernoulli-Euler theory flexural vibrations of beam, the effect of rotatory inertia and shear are neglected. However, the equations obtained on these assumptions are inadequate for short and thin-webbed beams and for beams where higher modes are required, considerable errors may be incurred by use of equations. When a beam is subjected to lateral vibration so that depth of the beam is a significant proportion of the distance between two adjacent nodes, rotatory inertia of beam and transverse deformation arising from the severe contortions of the beam during vibration make significant contributions to the lateral deflection. Therefore rotatory inertia and shear effects must be taken into account in the and analysis of high-frequency vibration of all beams, and in all analyses of deep beams. In this work, the effect of rotatory inertia and transverse-shear deformation on beams are introduced and analysed. Frequency equation for each case are developed in terms of dimensionless parameters of rotatory and shear. Numerical examples presented in literature are re-examined.*
Keywords: *beam theory, critical frequency, dispersion relation, rotatory inertia, shear deformation*

1. INTRODUCTION

Beam is often used as a structural element in many engineering structures, like frames, rotor blades and ship hulls, to model building behavior both for stability or dynamic analysis. Although, deflection of beams has interested the human since the century XVII. The first deflection studies were claimed by the Bernoulli family (Timoshenko, 1953). Daniel Bernoulli (1700-1782) and his pupil Leonhard Euler (1707-1783), continues the beam deflection studies of his uncle Jacob Bernoulli (1654-1705). Euler published in 1744 and 1773 the Euler-Bernoulli beam also known as the classical theory, with the suggestion gave by his advisor of applying the variational calculus to derive the equation of deflection curve. However, Saint Venant (1797-1886) noticed that the initial conditions for the beam problem needed to be reformulated, because due to increasing thickness, the plane sections do not remain plane and perpendicular. John William Strutt (1842-1919), known by Lord Rayleigh, in 1877 tried to resolve the inaccuracy noticed by his advisor, Saint Venant, considering the influence of the rotatory inertia. Rayleigh (1877a) analysed the beam by using the Hamilton's principle approach and in the discussion about waves in beams in terms of dispersion relation, phase and group speed. Due to new formulation occurred a slight improvement in the results for non-slender beams. Stephen Timoshenko (1878-1972) realized that it was necessary to considerate the shear deformation contribution for the formulation to become closer reality. Beam theory proposed by Timoshenko (1921) was in remarkably good agreement with the exact equations derived from the general equations of the theory of elasticity studied by Pochhammer (1876) and Chree (1889). Since Timoshenko published his beam model, many researchers investigated and improved the understand about beam behavior.

Goens (1931) obtained the solution of the differential equation of motion in terms of hyperbolic and trigonometric terms for the free-free boundary condition and noticed that from a critical frequency the hyperbolic term became trigonometric. Huang (1961) published the frequency equations and mode of vibration for the classical boundary conditions. In order to improve Timoshenko beam results for higher frequencies (frequencies above critical frequency), Cowper (1966) developed a general expression for shape factor based on elasticity. Thomas and Abbas (1975), Downs (1976) and Levinson and Cooke (1982) are credited for studies on the dynamic behavior of Timoshenko beams from higher frequencies. Han *et al.* (1999) was the first to present a wider study of beams in general by discussing the four theories (Euler-Bernoulli, Rayleigh, Shear and Timoshenko). Following Rayleigh's waves analysis, various authors obtained and discussed the dispersion relation, phase and group velocity for Timoshenko beams, some are Kolsky (1964), Wang and So (2005) and Smith (2008). This paper presents a review of principals beam theories. The motions equation, that describe the behaviour of a physical system in terms of its motion as a function of times, are derived from Euler-Lagrange equation. The effects of transverse shear deformation and rotatory inertia are included in the governing equations. Applying the adequate boundary conditions, frequency equations are obtained and their roots are calculated by using False Position Method. Numerical results shown to signify the differences in natural frequency, phase speed and group velocity prediction about each of four models.

2. EULER-BERNOULLI BEAM MODEL

In the Euler-Bernoulli theory or thin beam theory, the rotation of cross sections of the beam is neglected compared to the translation. In addition, the angular distortion due to shear is considered negligible compared to the bending deformation. The thin beam theory is applicable to beams for which the length is much larger than the depth (at least 10 times) and the deflections are small compared to the depth. The potential and kinetic energy for this model are given by Han *et al.* (1999):

$$U_b = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v(x,t)}{\partial x^2}\right)^2 dx \quad \text{and} \quad T_t = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial v(x,t)}{\partial t}\right)^2 dx, \tag{1}$$

where L is the length of beam, ρ, the mass per unit volume, A, the cross-sectional area, I, the moment of inertia of cross section, E, the modulus of elasticity, and $v(x,t)$ is the transverse deflection at the axial location x and time t. The governing partial differential equation of motion for free-vibration system is first given by (Euler, 1773):

$$d^4\,\frac{\partial^4 v(x,t)}{\partial x^4} - \frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{2}$$

where d is a coefficient relating acceleration and forces expressed as $\partial^4 v/\partial x$ and $\partial^2 v/\partial t^2$ respectively. However, Eq. (2) is commonly written as (Scarpello and Ritelli, 2003):

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} = 0. \tag{3}$$

Assume that the beam is excited harmonically with a frequency f and

$$v(x,t) = V(x)e^{jft} \quad \text{and} \quad \xi = x/L, \tag{4}$$

where $j = \sqrt{-1}$, ξ is the non-dimensional length of the beam and $V(x)$ is normal function of $v(x)$. Substituting the relations presented in Eq. (4) into Eq. (3) and omitting the common term e^{jft}

$$\frac{\partial^4 V(\xi)}{\partial \xi^4} - b^2 V(\xi) = 0, \quad \text{with} \quad b^2 = \frac{\rho A L^4}{EI}f^2, \quad \text{and} \quad f = 2\pi\omega, \tag{5}$$

where ω is natural frequency. The solution $V(\xi)$ can be expressed as follow:

$$V(\xi) = C_1 cos(\beta_e\xi) + C_2 sin(\beta_e\xi) + C_3 cosh(\beta_e\xi) + C_4 sinh(\beta_e\xi), \quad \text{where} \quad \beta_e = \sqrt{b}. \tag{6}$$

Function $V(\xi)$ is know as the normal mode or characteristic function of the beam. C_1, C_2, C_3 and C_4 are constants which can be found from the boundary conditions. In the Tab. 1 are showed the classical boundary conditions, the frequency equations and the first four values for b_i excluding the rigid-body mode ($b_i = 0$).

Table 1: *bi* eigenvalues and frequency equations of Euler-Bernoulli model.

	frequency equation	b_1	b_2	b_3	b_4
clamped-clamped	$cos\beta_e cosh\beta_e - 1 = 0$	22.3733	61.6728	120.9034	199.8594
hinged-hinged	$sin\beta_e sinh\beta_e = 0$	9.8696	39.4784	88.8264	157.9137
free-free	$cos\beta_e cosh\beta_e - 1 = 0$	22.3733	61.6728	120.9034	199.8594
clamped-free	$cos\beta_e cosh\beta_e + 1 = 0$	3.5160	22.0345	61.6972	120.9019

A special characteristic of this model is that the frequency equation for the free-free and clamped-clamped cases are the same. Frequency equations presented in Tab. 1 are transcendental and need to be solved numerically. In this paper was used the False Position root finding method. The values for b_i (with i=1,2,3,...) can be found from the appropriate frequency equation and then the natural frequencies ω_i (with i=1,2,3,...) may be found solving Eq. (5) for ω:

$$\omega_i = \frac{b_i}{2\pi}\sqrt{\left(\frac{EI}{\rho A L^4}\right)} \quad \text{with} \quad i = 1, 2, ..., n. \tag{7}$$

The general solution for propagation of waves along a beam can be assumed as Wang and So (2005):

$$v(x,t) = De^{j(kx - ft)}, \tag{8}$$

where D is the amplitude, k, the wave number, and f the frequency. The relation of the wave number k and its frequency f is called dispersion relation. Substituting Eq. (8) into Eq. (3), the dispersion relationship of the Euler-Bernoulli beam is given by:

$$f^2 = \frac{EI}{\rho A}k^4. \tag{9}$$

When a solid medium is deformed two types of elastic waves may be propagated: dilatation and distortion waves. Waves of dilatation propagates as compressional waves with the velocity c_l while waves of distortion propagates as transverse waves with the velocity c_t given by Kolsky (1964):

$$c_l^2 = (B + 4G/3)/\rho \quad \text{and} \quad c_t^2 = G/\rho, \tag{10}$$

where B is the bulk modulus and G the modulus of rigidity. The ratio of the velocities c_l and c_t can be presented as a function of the Poisson's ratio ν:

$$c_t = c_l\sqrt{\frac{1-2\nu}{2(1-\nu)}}. \tag{11}$$

Rayleigh (1877b) defines phase speed c and group velocities c_g of flexural waves propagating on a elastic medium as:

$$c = \frac{f}{k} \quad \text{and} \quad c_g = \frac{\partial f}{\partial k}. \tag{12}$$

Therefore, Eq. (12) can be rewritten as:

$$c = k\sqrt{\left(\frac{EI}{\rho A}\right)} \quad \text{and} \quad c_g = 2k\sqrt{\left(\frac{EI}{\rho A}\right)}. \tag{13}$$

3. RAYLEIGH BEAM MODEL

The fact that the phase speed and group speed presented in Eq. (13) are linearly increasing with the wave number k leads to infinity velocities of waves propagation as k approaches infinity. This physical incoherence was solved by Lord Rayleigh (1877a) by inclusion of the rotatory inertia effects to the classical theory. The kinetic energy due to the rotation of the cross section is given by:

$$T_r = \frac{1}{2}\int_0^L \rho I\left(\frac{\partial^2 v(x,t)}{\partial t\,\partial x}\right)^2 dx. \tag{14}$$

The kinetic energy of the beam is derived partly from the motion of translation (Eq. 1) and partly from the rotation about axes through their centres of inertia perpendicular to the plane of vibration (Eq. 14). The total kinetic energy is given by:

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial v(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L \rho I\left(\frac{\partial^2 v(x,t)}{\partial t\,\partial x}\right)^2 dx. \tag{15}$$

Equation of motion is obtained using Hamilton's principle:

$$\int_{t_1}^{t_2} \delta(T-U)\,dt + \int_{t_1}^{t_2} \delta W_{nc}\,dt = 0. \tag{16}$$

where T and U, are the the total kinetic and potential energy of the system respectively, δW_{nc} the virtual work done by nonconservative forces, t_1 and t_2, times at which the configuration of the system is known, and $\delta(\,)$ the symbol denoting virtual change, in the quantity in parentheses. Substituting Eq. (1) and Eq. (15) into Eq. (16) and assuming $\delta W_{nc} = 0$, equation of motion can be expressed as (Rayleigh, 1877a):

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} - \rho I\frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} = 0. \tag{17}$$

Assume that the beam is excited harmonically with a frequency f and adopting the same procedure presented in Eq. (4), we obtain:

$$\frac{\partial^4 V(\xi)}{\partial \xi^4} + b^2 r^2\frac{\partial^2 V(\xi)}{\partial \xi^2} - b^2 V(\xi) = 0, \quad \text{where} \quad r^2 = \frac{I}{AL^2} \tag{18}$$

is a coefficient related with the effect of rotatory inertia. The solution $V(\xi)$ can be expressed in terms of trigonometric and hyperbolic functions as follows:

$$V(\xi) = C_1 cosh(\alpha_r\xi) + C_2 sinh(\alpha_r\xi) + C_3 cos(\beta_r\xi) + C_4 sin(\beta_r\xi), \tag{19}$$

with

$$\alpha_r = \frac{b}{\sqrt{2}}\sqrt{-r^2 + \sqrt{r^4 + 4/b^2}} \qquad \text{and} \qquad \beta_r = \frac{b}{\sqrt{2}}\sqrt{r^2 + \sqrt{r^4 + 4/b^2}}. \tag{20}$$

Observe that Eq. (19) have two eigenvalues, α_r and β_r, that are related with trigonometric and hyperbolic sines and cosines respectively. Also observe that for $r = 0$ in Eq. (20), these eigenvalues turns to $\alpha_r = \beta_r = \beta_e$. Frequency equations for some boundary conditions are presented in Tab. 2.

Table 2: Frequency equations of Rayleigh model.

	frequency equation
clamped-clamped	$2 - 2\,cosh(\alpha_r)cos(\beta_r) - b\,r^2 sinh(\alpha_r)sin(\beta_r) = 0$
hinged-hinged	$sin(\beta_r)sinh(\alpha_r) = 0$
free-free	$2 - 2\,cosh(\alpha_r)cos(\beta_r) + b(b^2r^6 + 3r^2)sinh(\alpha_r)sin(\beta_r) = 0$
clamped-free	$2 + (b^2r^4 + 2)cosh(\alpha_r)cos(\beta_r) - b\,r^2 sinh(\alpha_r)sin(\beta_r) = 0$

Notice that as r approaches zero, the frequency equations presented in Tab. 2 becomes identical to that of Tab. 1. Unlike the Euler-Bernoulli beam case, natural frequency ω is written in terms of two eigenvalues (α_r and β_r) as follows:

$$\omega_i = \frac{\sqrt{\beta_r^2 - \alpha_r^2}}{2\pi r}\sqrt{\left(\frac{EI}{\rho A L^4}\right)} \quad \text{with} \quad i = 1, 2, ..., n. \tag{21}$$

Substituting Eq. (8) into Eq. (19), the dispersion relationship for Rayleigh beam is given by:

$$f^2 = \frac{EIk^4}{\rho A + \rho I k^2}. \tag{22}$$

Due to rotatory inertia effect, the phase speed c and group velocity c_g presented in Eq. (12) become:

$$c = k\sqrt{\left(\frac{EI}{\rho A + \rho I k^2}\right)} \quad \text{and} \quad c_g = 2c - \frac{\rho}{E}c^3. \tag{23}$$

4. SHEAR BEAM MODEL

A still more accurate differential equation is obtained if the deflection due to shear will be taken into account. This model adds the effect of shear distortion but not rotatory inertia to the Euler-Bernoulli model. The potential energy due to shear is given by (Timoshenko, 1921):

$$U_s = \frac{1}{2}\int_0^L KGA(\theta)^2 dx, \tag{24}$$

where K is the shape factor or shear coefficient, θ, the angle of rotation due to shear. Shape factor is a coefficient that multiplies the angle of shear θ at the neutral line of a beam to give an average value of θ for the entire cross-section. Timoshenko (1921) introduced K as a numerical factor depending on the shape of cross section. However, this definition leads to unsatisfactory results when Timoshenko beam equations are used to calculate the high frequency spectrum. Satisfactory results were obtained by Cowper (1966), he developes a general expression for K during the process of specialization of the equations of three-dimensional elasticity to the Timoshenko beam equation. In Cowper's definition K is a function of cross-section shape and Poisson's ratio ν given by:

$$K = \frac{4(1+\nu)I^2}{\nu\big(\int\int y^2 dxdy - I\big)I - 2A\int\int x\big(X(x,y) + xy^2\big)dxdy}, \tag{25}$$

where $X(x, y)$ is a harmonic function which satisfies the boundary of the cross section. Cowper calculates shear coefficient K for various cross sections, some of the values are tabulated in Tab. 3.

Table 3: The shear factor.

Cross section	K
Rectangle	$K = \dfrac{10(1+\nu)}{12 + 11\nu}$
Circle	$K = \dfrac{6(1+\nu)}{7 + 6\nu}$
Thin-walled round tube	$K = \dfrac{2(1+\nu)}{4 + 3\nu}$

Let $\partial v/\partial x$ denote the slope of the deflection curve when the shearing force is neglected and θ the angle of shear at the neutral axis in the same cross section, then we find for total slope:

$$\psi = \theta + \frac{\partial v}{\partial x}. \tag{26}$$

Note that the shear angle is neglected in Euler-Bernoulli and Rayleigh model, such as the bending rotation become equal to total rotation ($\psi = \partial v/\partial x$). Due to shear deformation effect addiction the potential energy due to bending given in Eq. (1) is slightly modified to include ψ:

$$U_b = \frac{1}{2}\int_0^L EI\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2 dx. \tag{27}$$

The potential energy of the beam is derived partly from the bending deformation (Eq. 27) and partly from the shear deformation (Eq. 24). Substituting Eq. (26) into Eq. (24), total potential energy is given by:

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L KGA(\psi - \frac{\partial v}{\partial x})^2 dx. \tag{28}$$

Substituting Eq. (1) and Eq. (28) on Hamilton's Principle (Eq. 16) we have two coupled equations expressed as:

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\rho EI}{KG}\frac{\partial^4 v(x,t)}{\partial x^2\partial t^2} = 0, \tag{29}$$

$$EI\frac{\partial^4 \psi(x,t)}{\partial x^4} + \rho A\frac{\partial^2 \psi(x,t)}{\partial t^2} - \frac{\rho EI}{KG}\frac{\partial^4 \psi(x,t)}{\partial x^2\partial t^2} = 0. \tag{30}$$

Unlike in the Euler-Bernoulli beam and the Rayleigh models, there are two dependent variables (v and ψ) for Shear beam. Assume that the beam is excited harmonically with a frequency f and

$$v(x,t) = V(x)e^{jft}, \quad \psi(x,t) = \Psi(x)e^{jft} \quad \text{and} \quad \xi = x/L. \tag{31}$$

Substituting the relations presented in Eq. (31) into Eqs. (29-30) and adopting the same procedure as before, we obtain:

$$\frac{\partial^4 V(\xi)}{\partial\xi^4} + b^2 s^2\frac{\partial^2 V(\xi)}{\partial\xi^2} - b^2 V(\xi) = 0, \quad \text{and} \quad \frac{\partial^4 \Psi(\xi)}{\partial\xi^4} + b^2 s^2\frac{\partial^2 \Psi(\xi)}{\partial\xi^2} - b^2 \Psi(\xi) = 0, \tag{32}$$

where

$$s^2 = \frac{EI}{KGAL^2} \quad \text{or} \quad s = r\,\lambda, \quad \text{with} \quad \lambda = \sqrt{\frac{E}{KG}}, \tag{33}$$

s is a coefficient related with the effect of shear distortion, and λ represents the relation between r and s. The solution $V(\xi)$ and $\Psi(\xi)$ can be expressed as:

$$V(\xi) = C_1 cosh(\alpha_s\xi) + C_2 sinh(\alpha_s\xi) + C_3 cos(\beta_s\xi) + C_4 sin(\beta_s\xi), \tag{34}$$

$$\Psi(\xi) = C_1' sinh(\alpha_s\xi) + C_2' cosh(\alpha_s\xi) + C_3' sin(\beta_s\xi) + C_4' cos(\beta_s\xi), \tag{35}$$

with

$$\alpha_s = \frac{b}{\sqrt{2}}\sqrt{-s^2 + \sqrt{s^4 + 4/b^2}} \quad \text{and} \quad \beta_s = \frac{b}{\sqrt{2}}\sqrt{s^2 + \sqrt{s^4 + 4/b^2}}. \tag{36}$$

Equations (34) and (35) have two eigenvalues, α_s and β_s, that are related with trigonometric and hyperbolic sines and cosines respectively. Note that similarly as Rayleigh model, for $s = 0$ in Eq. (36), these eigenvalues turns to $\alpha_s = \beta_s = \beta_e$. The constants of Eq. (34) and Eq. (35) are related as follows:

$$C_1' = \frac{b^2(\alpha_s^2 + s^2)}{\alpha_s L}C_1, \quad C_2' = \frac{b^2(\alpha_s^2 + s^2)}{\alpha_s L}C_2, \quad C_3' = -\frac{b^2(\beta_s^2 - s^2)}{\beta_s L}C_3 \quad \text{and} \quad C_4' = \frac{b^2(\beta_s^2 - s^2)}{\beta_s L}C_4. \tag{37}$$

Shear model frequency equations for hinged-hinged, clamped-clamped, free-free and clamped-free boundary conditions are presented in Tab. 4. Observe that as s approaches zero, the frequency equations presented in Tab. 4 becomes identical to that of Tab. 1. The natural frequency ω is written in terms of two eigenvalues (α_s and β_s) as follows:

$$\omega_i = \frac{\sqrt{\beta_s^2 - \alpha_s^2}}{2\pi s}\sqrt{\left(\frac{EI}{\rho AL^4}\right)} \quad \text{with} \quad i = 1, 2, ..., n. \tag{38}$$

Substituting Eq. (8) into Eq. (29), the dispersion relationship for Shear beam is given by:

$$f^2 = \frac{EIk^4}{\rho A + \frac{\rho EI}{KG}k^2}. \tag{39}$$

Due to shear distortion effect, the phase speed c and group velocity c_g presented in Eq. (12) become:

$$c = k\sqrt{\left(\frac{EI}{\rho A + \frac{\rho EI}{KG}k^2}\right)} \quad \text{and} \quad c_g = 2c - \frac{\rho}{KG}c^3. \tag{40}$$

Table 4: Frequency equations of Shear model.

	frequency equation
clamped-clamped	$2 - 2\,cosh(\alpha_s)cos(\beta_s) + b(b^2s^6 + 3s^2)sinh(\alpha_s)sin(\beta_s) = 0$
hinged-hinged	$sin(\beta_s)sinh(\alpha_s) = 0$
free-free	$2 - 2\,cosh(\alpha_s)cos(\beta_s) - b(s^2)sinh(\alpha_s)sin(\beta_s) = 0$
clamped-free	$2 + (b^2(s^4) + 2)cosh(\alpha_s)cos(\beta_s) - bs^2sinh(\alpha_s)sin(\beta_s) = 0$

5. TIMOSHENKO BEAM MODEL

Timoshenko (1921) proposed a beam model which includes both rotatory inertia and shear deformation effects to classical theory. Timoshenko theory is a major improvement for non-slender beams and for high-frequency responses where shear or rotary effects are not negligible. Similar to Shear model, the potential energy of the beam is derived partly from the bending deformation and partly from the shear deformation. Therefore, total potential energy is given by:

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L KGA(\psi - \frac{\partial v}{\partial x})^2 dx. \tag{41}$$

Similar to Rayleigh model, the kinetic energy of the beam is derived partly from the motion of translation and partly from the rotation. However, due to shear effect addiction the kinetic energy due to rotation given in Eq. (14) is slightly modified to include ψ:

$$T_r = \frac{1}{2}\int_0^L \rho I\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2 dx. \tag{42}$$

Therefore, total kinetic energy is given by:

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial v(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L \rho I\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2 dx. \tag{43}$$

Substituting Eq. (41) and Eq. (43) on Hamilton's Principle (Eq. 16) we have two coupled equations expressed as:

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\rho EI}{kG}\frac{\partial^4 v(x,t)}{\partial t^2\partial x^2} - \rho I\frac{\partial^4 v(x,t)}{\partial x^2\partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 v(x,t)}{\partial t^4} = 0, \tag{44}$$

$$EI\frac{\partial^4 \psi(x,t)}{\partial x^4} + \rho A\frac{\partial^2 \psi(x,t)}{\partial t^2} - \frac{\rho EI}{kG}\frac{\partial^4 \psi(x,t)}{\partial t^2\partial x^2} - \rho I\frac{\partial^4 \psi(x,t)}{\partial x^2\partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 \psi(x,t)}{\partial t^4} = 0. \tag{45}$$

Assume that the beam is excited harmonically with a frequency f and adopting the same procedure presented in Eq. (31), we obtain (Soares and Hoefel, 2015):

$$\frac{\partial^4 V(\xi)}{\partial\xi^4} + b^2s^2\frac{\partial^2 V(\xi)}{\partial\xi^2} + b^2r^2\frac{\partial^2 V(\xi)}{\partial\xi^2} + b^4r^2s^2V(\xi) - b^2V(\xi) = 0, \tag{46}$$

$$\frac{\partial^4 \Psi(\xi)}{\partial\xi^4} + b^2s^2\frac{\partial^2 \Psi(\xi)}{\partial\xi^2} + b^2r^2\frac{\partial^2 \Psi(\xi)}{\partial\xi^2} + b^4r^2s^2\Psi(\xi) - b^2\Psi(\xi) = 0. \tag{47}$$

We must consider two cases when obtaining Timoshenko beam model spatial solution (Han *et al.*, 1999). In the first case, assume:

$$\sqrt{(r^2 - s^2)^2 + 4/b^2} > (r^2 + s^2) \quad \text{which leads to} \quad b < \frac{1}{(r\,s)}, \tag{48}$$

while in the second

$$\sqrt{(r^2 - s^2)^2 + 4/b^2} < (r^2 + s^2) \quad \text{which leads to} \quad b > \frac{1}{(r\,s)}. \tag{49}$$

Substituting $b = 1/(rs)$ in Eq. (5), we have the critical frequency expressed as:

$$f_c = \sqrt{\frac{KGA}{\rho I}} \quad \text{or} \quad \omega_c = \frac{\sqrt{KGA/\rho I}}{2\pi}. \tag{50}$$

We call this cutoff value $b_{crit} = 1/(rs)$. When $b < b_{crit}$ the solution $V(\xi)$ and $\Psi(\xi)$ can be expressed in sinusoidal and hyperbolic terms:

$$V(\xi) = C_1cosh(\alpha_{t1}\xi) + C_2sinh(\alpha_{t1}\xi) + C_3cos(\beta_t\xi) + C_4sin(\beta_t\xi), \tag{51}$$

$$\Psi(\xi) = C_1'sinh(\alpha_{t1}\xi) + C_2'cosh(\alpha_{t1}\xi) + C_3'sin(\beta_t\xi) + C_4'cos(\beta_t\xi), \tag{52}$$

with

$$\alpha_{t1} = \frac{b}{\sqrt{2}}\sqrt{-(r^2+s^2)+\sqrt{(r^2-s^2)^2+\frac{4}{b^2}}} \quad \text{and} \quad \beta_t = \frac{b}{\sqrt{2}}\sqrt{(r^2+s^2)+\sqrt{(r^2-s^2)^2+\frac{4}{b^2}}}. \tag{53}$$

Equations (51) and (52) have two eigenvalues, α_{t1} and β_t, that are related with trigonometric and hyperbolic sines and cosines respectively. Note that the coefficients r and s relates the four theories of beam. On Timoshenko model $r = 0$ leads to $\alpha_{t1} = \alpha_s$ and $\beta_t = \beta_s$, for $s = 0$ the eigenvalues turn to $\alpha_{t1} = \alpha_r$ and $\beta_t = \beta_r$, finally for $r = s = 0$ we have $\alpha_{t1} = \beta_t = \beta_e$. When $b > b_{crit}$ the solution $V(\xi)$ and $\Psi(\xi)$ can be expressed only in sinusoidal terms:

$$V(\xi) = \overline{C_1}cos(\alpha_{t2}\xi) + \overline{C_2}sin(\alpha_{t2}\xi) + \overline{C_3}cos(\beta_t\xi) + \overline{C_4}sin(\beta_t\xi), \tag{54}$$

$$\Psi(\xi) = \overline{C_1}'sin(\alpha_{t2}\xi) + \overline{C_2}'cos(\alpha_{t2}\xi) + \overline{C_3}'sin(\beta_t\xi) + \overline{C_4}'cos(\beta_t\xi), \tag{55}$$

with

$$\alpha_{t2'} = j\frac{b}{\sqrt{2}}\sqrt{(r^2+s^2)-\sqrt{(r^2-s^2)^2+\frac{4}{b^2}}} = j\alpha_{t2} \quad \text{and} \quad \beta_t = \frac{b}{\sqrt{2}}\sqrt{(r^2+s^2)+\sqrt{(r^2-s^2)^2+\frac{4}{b^2}}}. \tag{56}$$

Equations (54) and (55) have two eigenvalues α_{t2} and β_t. Notice that $\alpha_{t2'}$ values are always complex. The relations between the coefficients in Eqs. (51) and (52), or Eqs. (54) and (55) are given by (Huang, 1961):

$$C_1' = \frac{\alpha_{t1}^2 + b^2s^2}{\alpha_{t1}L}C_1, \quad C_2' = \frac{\alpha_{t1}^2 + b^2s^2}{\alpha_{t1}L}C_2, \quad C_3' = -\frac{\beta_t^2 - b^2s^2}{\beta_t L}C_3, \quad \text{and} \quad C_4' = \frac{\beta_t^2 - b^2s^2}{\beta_t L}C_4, \tag{57}$$

$$\overline{C_1}' = -\frac{\alpha_{t2}^2 - b^2s^2}{\alpha_{t1}L}\overline{C_1}, \quad \overline{C_2}' = \frac{\alpha_{t2}^2 - b^2s^2}{\alpha_{t2}L}\overline{C_2}, \quad \overline{C_3}' = -\frac{\beta_t^2 - b^2s^2}{\beta_t L}\overline{C_3}, \quad \text{and} \quad \overline{C_4}' = \frac{\beta_t^2 - b^2s^2}{\beta_t L}C_4. \tag{58}$$

Table 5: Frequency equations of Timoshenko model.

	frequency equation ($b < b_{crit}$)
c-c	$2 - 2cosh(\alpha_{t1})cos(\beta_t) + \frac{b(b^2s^2(r^2-s^2)^2+(3s^2-r^2))}{\sqrt{1-b^2r^2s^2}}sinh(\alpha_{t1})sin(\beta_t) = 0$
h-h	$sin(\beta_t)sinh(\alpha_{t1}) = 0$
f-f	$2 - 2cosh(\alpha_{t1})cos(\beta_t) + \frac{b(b^2r^2(r^2-s^2)^2+(3r^2-s^2))}{\sqrt{1-b^2r^2s^2}}sinh(\alpha_{t1})sin(\beta_t) = 0$
c-f	$2 + (b^2(r^2-s^2)^2+2)cosh(\alpha_{t1})cos(\beta_t) - \frac{b(r^2+s^2)}{\sqrt{1-b^2r^2s^2}}sinh(\alpha_{t1})sin(\beta_t) = 0$
	frequency equation ($b > b_{crit}$)
c-c	$2 - 2cos(\alpha_{t2})cos(\beta_t) + \frac{b(b^2s^2(r^2-s^2)^2+(3s^2-r^2))}{\sqrt{b^2r^2s^2-1}}sin(\alpha_{t2})sin(\beta_t) = 0$
h-h	$sin(\alpha_{t2})sin(\beta_t) = 0$
f-f	$2 - 2cos(\alpha_{t2})cos(\beta_t) + \frac{b(b^2r^2(r^2-s^2)^2+(3r^2-s^2))}{\sqrt{b^2r^2s^2-1}}sin(\alpha_{t2})sin(\beta_t) = 0$
c-f	$2 + (b^2(r^2-s^2)^2+2)cos(\alpha_{t2})cos(\beta_t) - \frac{b(r^2+s^2)}{\sqrt{b^2r^2s^2-1}}sin(\alpha_{t2})sin(\beta_t) = 0$

Timoshenko model have two pair of frequency equations for each classic boundary conditions. Frequency equations when $b < b_{crit}$ and $b > b_{crit}$ are presented in Tab. 5 for clamped-clamped (c-c), hinged-hinged (h-h), free-free (f-f), and clamped-free (c-f) boundary conditions. Note that as r and s approaches zero, the frequency equations presented in Tab. 5 becomes identical to that of Tab. 1 and b_{crit} approaches infinity. It can be observed in Eq. (5) that if b_{crit} approaches infinity, ω_{crit} can no longer be defined, in fact ω_{crit} only appears if both rotatory inertia and shear deformation effects are considered. The natural frequency ω is written in terms of two eigenvalues (α_{t1} and β_t or α_{t2} and β_t) as follows (Huang, 1961):

$$\omega_i = \frac{\sqrt{\beta_t^2 - \alpha_{t1}^2}}{2\pi\sqrt{r^2+s^2}}\sqrt{\left(\frac{EI}{\rho AL^4}\right)} \quad \text{with} \quad i = 1, 2, ..., n. \quad \text{when} \quad b < b_{crit}, \tag{59}$$

$$\omega_i = \frac{\sqrt{\beta_t^2 + \alpha_{t2}^2}}{2\pi\sqrt{r^2+s^2}}\sqrt{\left(\frac{EI}{\rho AL^4}\right)} \quad \text{with} \quad i = 1, 2, ..., n. \quad \text{when} \quad b > b_{crit}. \tag{60}$$

Substituting Eq. (8) into Eq. (44), the dispersion relationship for Timoshenko beam is given by:

$$f^2 = \frac{KG}{2\rho^2 I}\left(\rho A + \rho I k^2 + \frac{\rho EI}{KG}k^2 \pm \sqrt{\left(-\rho A - \rho I k^2 - \frac{\rho EI}{KG}k^2\right)^2 - 4EIk^4\frac{\rho^2 I}{KG}}\right). \tag{61}$$

Due to both shear distortion and rotatory inertia effects, the phase speed c and group velocity c_g presented in Eq. (12) become:

$$c = \sqrt{\frac{KG}{2\rho^2 I k^2}\left(\rho A + \rho I k^2 + \frac{\rho EI}{KG}k^2 \pm \sqrt{\left(-\rho A - \rho I k^2 - \frac{\rho EI}{KG}k^2\right)^2 - 4EIk^4\frac{\rho^2 I}{KG}}\right)}, \tag{62}$$

$$c_g = \frac{1}{2c}\frac{KG}{\rho^2 I}\left[\rho I + \frac{\rho EI}{KG} \pm \frac{\left(\rho I + \frac{\rho EI}{KG}\right)\rho A + \left(\left(\rho I + \frac{\rho EI}{KG}\right)^2 + \frac{4E\rho^2 I^2}{KG}\right)k^2}{\sqrt{\left(-\rho A - \left(\rho I + \frac{\rho EI}{KG}\right)k^2\right)^2 - \frac{4E\rho^2 I^2}{KG}k^4}}\right]. \tag{63}$$

6. NUMERICAL RESULTS

Solutions of frequency equations given in Tabs. 1, 2, 4 and 5 are presented as a continuous function of r or s. In order to solve frequency equations was used the False Position root finding method. Results for clamped-clamped beam are shown in Figs. 1 and 2, similar plots are given in Han *et al.* (1999). Figure 1 presents the first four Rayleigh and

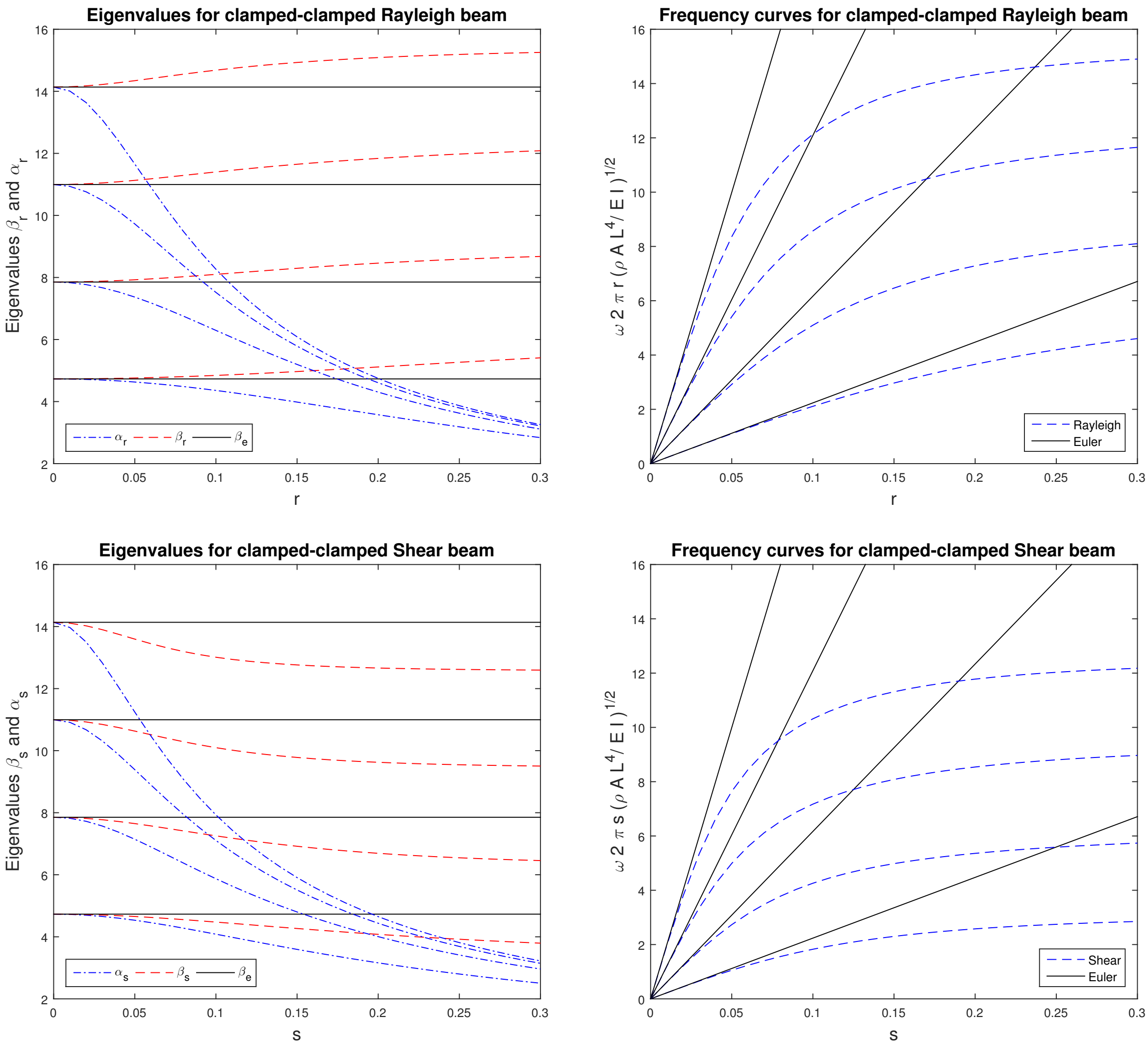

Figure 1: Eigenvalues and frequency curves for clamped-clamped Rayleigh and Shear beam.

Shear eigenvalues and natural frequencies as functions of r and s respectively. The four solid lines are the first four Euler-Bernoulli eigenvalues (β_e), observe that they are not affected by parameters r and s unlike the pairs: α_r and β_r, and α_s

and β_s. Note that besides Euler-Bernoulli theory predicts a linear increasing to frequencies values as given parameters increases, Rayleigh and Shear theories are very similar to classical model for slender beams. However, frequency curves in Fig. 1 shows that, as r or s increases, shear deformation effect predicts better results than rotatory inertia. The proper way to represent the eigenvalues on Timoshenko model is to make a three-dimensional plot of α_t and β_t as functions of r and s simultaneously. However, due to the eigenvalues plots of previous beam models could only be presented in two-dimensional plot, Timoshenko eigenvalues and frequency curves are plotted as functions of r by relating the parameters using Eq. 33. Consider a clamped-clamped beam made of steel whose properties are given in Tab. 6 (Han *et al.*, 1999) .

Table 6: Properties of the beam.

$E\,(Gpa)$	$G\,(Gpa)$	$\rho\,(Kg/m^3)$	$A\,(m^2)$	$I\,(m^4)$	K	$L\,(m)$	$\lambda = \sqrt{E/(K\,G)}$
200	77.5	7830	0.0097389	0.0001171	0.53066	1	2.205

Timoshenko eigenvalues and frequency curves are plotted for $\lambda = 2.205$ in Fig. 2.

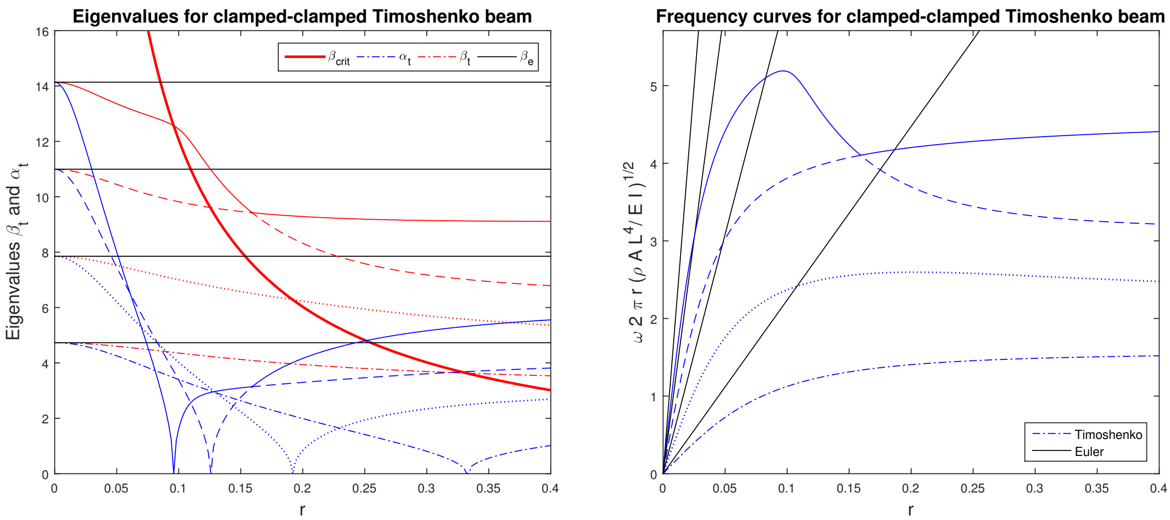

Figure 2: Eigenvalues and frequency curves for clamped-clamped Timoshenko beam.

Figure 2 presents eigenvalues and frequency to $b < b_{crit}$ (Eqs. (53) and (59)) and $b > b_{crit}$ (Eqs. (56) and (60)). The case $b = b_{crit}$ has been discussed by Levinson and Cooke (1982) who identifies the motion for this case as a shear oscillation without transverse deflection. When $b = b_{crit}$, α_{t1} and α_{t2} are equal to zero, and $\beta_t = \beta_{crit}$ as shown in Fig. 2. It is interesting to observe that eigenvalues curves were plotted with four line styles to shown that for $b > b_{crit}$, α_t and β_t third (dashed-dashed) and fourth (solid-line) curves do not cross each other. Frequency curves were also plotted with four line styles to shown that third (dashed-dashed) and fourth (solid-line) curves do not cross each other.

Although longitudinal and transverse velocity of waves are given by Eq. 10, the propagation of waves in a bounded elastic medium are slower and expressed with a different nomenclature: stress waves and shear waves. Stress waves propagates as compressional waves with the velocity v_l while shear waves propagates as transverse waves with the velocity v_t given by (Smith, 2008):

$$v_l = \sqrt{\frac{E}{\rho}} \quad \text{and} \quad v_t = \sqrt{\frac{KG}{\rho}}. \tag{64}$$

A comparison of the presented beam theories in terms of phase speed and group velocity are presented in Fig. 3 using the values given in Tab. 6. Phase speed and group velocity are plotted in function of k. Note that for higher wave numbers Shear and Rayleigh model approaches the shear and stress wave speed respectively, while as observed by Rayleigh Euler-Bernoulli model approaches infinity. However, due to both assumptions for rotatory inertia and shear deformation effects, only Timoshenko theory is able to predict two waves propagating in a beam.

7. CONCLUSION

This paper discusses the effects of rotatory inertia and shear deformation on beam theory. Frequency equations, phase speed and group velocity were obtained for classic boundary conditions. The differences between these effects were discussed by analysis of natural frequency and wave propagation results for slender and non-slender beams. It was observed that critical frequency occurs only on Timoshenko beam theory, due to both effects inclusion, and that eigenvalues turns to complex values for frequencies above critical frequency. The consequences for neglecting both effects on a beam was discussed and it was confirmed that Euler-Bernoulli model was inadequate for higher modes

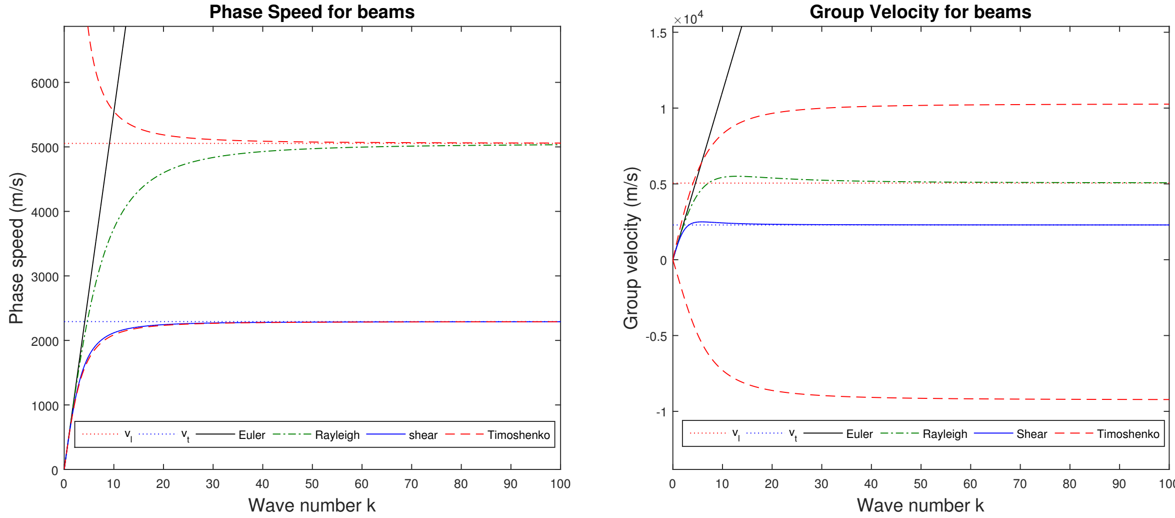

Figure 3: Phase speed and group velocity of a beam.

and thick beams. Also a brief review of researchers improvements along the centuries since Euler-Bernoulli model was published were presented. It was observed on a numerical example that shear deformation effect predicts better results than rotatory inertia for non-slender beams, although their results are very similar to classical theory for slender beams. However, a major improvement is obtained by the addiction of both effects on Euler-Bernoulli theory.

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9. RESPONSIBILITY NOTICE

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