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2 authors:

Anderson Soares Da Costa Azevêdo

Simone dos Santos
Universidade Federal do Piauí

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MODAL ANALYSIS FOR FREE VIBRATION OF FOUR BEAM THEORIES

Anderson Soares da Costa Azevêdo
Universidade Federal do Piauí
anderson4r2@hotmail.com

Simone dos Santos Hoefel
Universidade Federal do Piauí
simone.santos@ufpi.edu.br

Abstract. *Beams are structural elements frequently used for support buildings, part of airplanes, ships, rotor blades and most engineering structures. In many projects it is assumed that these elements are subjected only to static loads, however dynamic loads induce vibrations, which changes the values of stresses and strains. Furthermore, these mechanical phenomenon cause noise, instabilities and may also develop resonance, which improves deflections and failure. Therefore study these structural behavior is fundamental in order to prevent the effects of vibration. The mechanical behavior of these structural elements may be described by differential equations, four widely used beam theories are Euler-Bernouli, Rayleigh, Shear and Timoshenko. This paper presents a comparison between these four beam theories for the free transverse vibration of uniform beam. Numerical examples are shown to beam models under classical boundary condition.*

Keywords: *Euler-Bernoulli, Timoshenko, Rayleigh, Shear, Free Vibration*

1. INTRODUCTION

In the Euler-Bernoulli theory (Euler and Bousquet, 1744) , sometimes called the classical beam theory, of flexural vibrations of beams, the effects of rotatory inertia and shear are neglected. The equations obtained on these assumptions are adequate for relatively slender beams of lower modes (Wang, 1970). However, the Euler-Bernoulli beam theory tends to slightly overestimate the natural frequencies. It can be considered inadequate for those beams when the effect of the cross-sectional dimensions on frequencies cannot be neglected. The prediction is better for slender beams.

The Shear model (Han *et al.*, 1999) adds shear distortion to the Euler-Bernoulli model.This model is different from the pure shear model which includes the shear distortion and rotary inertia only or the simple shear beam which includes the shear distortion and lateral displacement only. By adding shear distortion to the Euler-Bernoulli beam, the estimate of the natural frequencies improves considerably. Neither the pure shear nor the model fits our purpose of obtaining an improved model to the Euler model because both exclude the most important factor, the bending effect. The Rayleigh beam theory (Rayleigh, 1894) provides a marginal improvement on the classical beam theory by including the effect of rotation of the cross-section. As a result, it partially corrects the overestimation of natural frequencies Euler-Bernoulli model. However, the natural frequencies are still overestimated. Timoshenko (Timoshenko, 1921) extend this to include the effect of shear as well as the effect of rotation to the Euler-Bernoulli beam. The Timoshenko model is a major improvement for non-slender beams and for high-frequency responses where shear or rotatory effects are not negligible. Several authors have obtained the frequency equations and the mode shapes for various boundary conditions. Some are Han *et al.* (1999), Grant (1978), Huang (1961).

In this paper, the partial differential equation of motion for each model is solved in full obtaining the frequency equations for each end condition, the solutions of these frequency equations in terms of dimensionless wave numbers, the orthogonality conditions among the eigenfunctions, and the procedure to obtain the full solution to the non-homogeneous initial-boundary-value problem using the method of Eigenfunction Expansion (Meirovitch, 2001). A numerical example is shown for non-slender beam to signify the differences among the four beam models.

2. EULER-BERNOULLI BEAM MODEL

The governing partial differential equation of motion for free-vibration system is given by (Rao, 2011):

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{1}$$

where ρ is the mass per unit volume, A is the cross-sectional area, E is the modulus of elasticity, I the moment of inertia of the cross-section and $v(x,t)$ is the transverse deflection at the axial location x and time t. The equation of motion, boundary conditions, and initial conditions form an initial-boundary-value problem which can be solved using methods of *separation of variables and eigenfunction expansion* (Han *et al.*, 1999). By separating $v(x,t)$ into two functions such that $v(x,t) = V(x) \times T(t)$, the Eq. (1) can be expressed as:

$$c^2 \times \frac{1}{V(x)} \frac{\partial^4 V(x)}{\partial x^4} = -\frac{1}{T(t)} \frac{\partial^2 T(t)}{\partial t^2}, \tag{2}$$

where $c = \sqrt{EI/(\rho A)}$, and $V(x)$ is known as the modal shapes of beam. The solutions of $T(t)$ and $V(x)$ can be expressed as follows:

$$T(t) = A_t \times cos(\omega t - \theta), \tag{3}$$

$$V(x) = C_1 sin(\beta x) + C_2 cos(\beta x) + C_3 sinh(\beta x) + C_4 cosh(\beta x). \tag{4}$$

Function $V(x)$ is know as the normal mode or characteristic function of the beam. A_t is constant, θ is the phase angle, C_1, C_2, C_3 and C_4, in each case, are different constants, which can be found from the boundary conditions. ω is the natural frequency and can be computed as:

$$\omega = \frac{(\beta L)^2 c}{L^2} \qquad \text{and} \qquad \beta = \sqrt[2]{\frac{\omega}{c}}. \tag{5}$$

For suppress C_n is necessary to normalize the vector modal shapes. The process of rendering the amplitude of a mode to be unique is called normalization, and the resulting modes, are called normal modes (Craig, 1981).

3. TIMOSHENKO BEAM MODEL

The governing coupled differential equations for transverse vibrations of Timoshenko beams are (Timoshenko, 1921):

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} - \frac{\rho EI}{kG}\frac{\partial^4 v}{\partial x^2 t^2} - \rho I\frac{\partial^4 v}{\partial x^2 t^2} + \frac{\rho I^2}{kG}\frac{\partial^4 v}{\partial t^4} = 0, \tag{6}$$

$$EI\frac{\partial^4 \psi}{\partial x^4} + \rho A\frac{\partial^2 \psi}{\partial t^2} - \frac{\rho EI}{kG}\frac{\partial^4 \psi}{\partial x^2 t^2} - \rho I\frac{\partial^4 \psi}{\partial x^2 t^2} + \frac{\rho I^2}{kG}\frac{\partial^4 \psi}{\partial t^4} = 0, \tag{7}$$

in which E is the modulus of elasticity, I, the moment of inertia of cross section, k, the shear coefficient, A, the cross-sectional area, G, the modulus of rigidity, ρ the mass per unit volume, v, the transverse deflection, and ψ the bending slope. Assume that the beam is excited harmonically with a frequency f and

$$v(x,t) = V(x)e^{jft}, \qquad \psi(x,t) = \Psi(x)e^{jft} \quad \text{and} \quad \xi = x/L, \tag{8}$$

where $j = \sqrt{-1}$, ξ is the non-dimensional length of the beam, $V(x)$ is normal function of $v(x)$, $\Psi(x)$ is normal function of ψ, and L, the length of the beam. Substituting the above relations into Eq. (6) and Eq. (7) through Eq. (8) and omitting the common term e^{jft}, the following equations are obtained

$$\frac{d^4 V}{d\xi^4} + b^2(r^2+s^2)\frac{d^2 V}{d\xi^2} - b^2(1 - b^2 r^2 s^2)V = 0, \tag{9}$$

$$\frac{d^4 \Psi}{d\xi^4} + b^2(r^2+s^2)\frac{d^2 \Psi}{d\xi^2} - b^2(1 - b^2 r^2 s^2)\Psi = 0, \tag{10}$$

where

$$b^2 = \frac{\rho A L^4}{EI} f^2 \qquad \text{with} \qquad f = 2\pi\omega, \tag{11}$$

where f is angular frequency, and ω, the natural frequency, and

$$r^2 = \frac{I}{AL^2} \qquad \text{and} \qquad s^2 = \frac{EI}{kAGL^2}, \tag{12}$$

are coefficients related with the effect of rotatory inertia and shear deformation. The solutions of equations Eq. (9) and Eq. (10) may be written as Huang (1961):

$$V(\xi) = C_1 cosh(b\alpha\xi) + C_2 sinh(b\alpha\xi) + C_3 cos(b\beta\xi) + C_4 sin(b\beta\xi), \tag{13}$$

$$\Psi(\xi) = C_1' sinh(b\alpha\xi) + C_2' cosh(b\alpha\xi) + C_3' sin(b\beta\xi) + C_4' cos(b\beta\xi), \tag{14}$$

where the function $V(\xi)$ is know as the normal mode of the beam, C_i and C_i', with $i = 1,2,3,4$, are coefficients which can be found from boundary conditions, and α and β are coefficients given as:

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{-(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}}} \qquad \text{and} \qquad \beta = \frac{1}{\sqrt{2}}\sqrt{(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}}}. \tag{15}$$

4. RAYLEIGH BEAM MODEL

As mentioned before, Rayleigh beam model adds the rotatory inertia effect to Euler-Bernoulli beam. For free vibration, the equation of motion can be expressed as (Han *et al.*, 1999):

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} - \rho I\frac{\partial^4 v}{\partial x^2 t^2} = 0. \tag{16}$$

Adopting the same procedure as before , we obtain the following vibration mode:

$$V(x) = C_1 cosh(b\alpha x) + C_2 sinh(b\alpha x) + C_3 cos(b\beta x) + C_4 sin(b\beta x), \tag{17}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{-(r^2) + \sqrt{r^4 + \frac{4}{b^2}}} \qquad \text{and} \qquad \beta = \frac{1}{\sqrt{2}}\sqrt{(r^2) + \sqrt{r^4 + \frac{4}{b^2}}}. \tag{18}$$

5. THE SHEAR BEAM MODEL

Shear beam model adds the the effect of shear distortion to the Euler-Bernoulli model. Unlike in the Euler-Bernoulli and Rayleigh beam models, there are two dependent variables for the shear beam. The equations of motion, are given by (Han *et al.*, 1999):

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} - \frac{\rho EI}{kG}\frac{\partial^4 v}{\partial x^2 t^2} = 0, \tag{19}$$

$$EI\frac{\partial^4 \psi}{\partial x^4} + \rho A\frac{\partial^2 \psi}{\partial t^2} - \frac{\rho EI}{kG}\frac{\partial^4 \psi}{\partial x^2 t^2} = 0. \tag{20}$$

Adopting the same procedure as before , we obtain the following vibration mode:

$$V(x) = C_1 cosh(b\alpha x) + C_2 sinh(b\alpha x) + C_3 cos(b\beta x) + C_4 sin(b\beta x), \tag{21}$$

$$\Psi(x) = C_1' sinh(b\alpha x) + C_2' cosh(b\alpha x) + C_3' sin(b\beta x) + C_4' cos(b\beta x), \tag{22}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{-(s^2) + \sqrt{s^4 + \frac{4}{b^2}}} \qquad \text{and} \qquad \beta = \frac{1}{\sqrt{2}}\sqrt{(s^2) + \sqrt{s^4 + \frac{4}{b^2}}}. \tag{23}$$

6. FREQUENCY EQUATION

The application of appropriate boundary conditions and relations of integration constants to equations Eq. (13) and Eq. (14) yields for each type of beam a set four homogeneous linear algebraic equations in four constants C_1 to C_4 with without primes (Stephen, 1980). In order that solution other than zero may exist the determinant of the coefficients C_4 must be equal to zero. This leads to the frequency equation in each case from which the natural frequencies can be determined. In this paper two cases to be considered: clamped-free and clamped-clamped. The necessary and sufficient boundary conditions for the beams are found as follows:

$$\text{Clamped end:} \quad V = 0, \quad \frac{d\Psi}{dx} = 0; \tag{24}$$

$$\text{Free end:} \quad \frac{d\Psi}{dx} = 0, \quad \frac{1}{L}\frac{dV}{dx} - \Psi = 0. \tag{25}$$

The frequency equations for clamped-clamped and clamped-free are as follows:

$$\text{Clamped-free beam:} \qquad 2 + \left[b^2(r^2 - s^2) + 2\right] cosh(b\alpha)cos(b\beta) - \frac{b(r^2 + s^2)}{(1 - b^2 r^2 s^2)^{1/2}} sinh(b\alpha)sin(b\beta) = 0, \tag{26}$$

$$\text{Clamped-clamped beam:} \qquad 2 - 2cosh(b\alpha)cos(b\beta) + \frac{b}{(1 - b^2 r^2 s^2)^{1/2}} \left[b^2 s^2 (r^2 - s^2)^2 + (3s^2 - r^2)\right] sinh(b\alpha)sin(b\beta) = 0. \tag{27}$$

7. NUMERICAL RESULTS

7.1 Clamped-clamped Beam

Consider a non-slender clamped-clamped beam showed in Fig (1) with $H/L = 0.277128$, and $L = 1m$. The values of $b_i(i = 1, 2, 3, ...)$ can be found from the appropriate frequency equations and then the natural frequencies $\omega_i(i = 1, 2, 3, ...)$ can be found from Eq. (11). These frequency equations are highly transcendental and are solved using Newton-Raphson method (Pradhan (2012), Pradhan and Mandal (2013), Gunda *et al.* (2009)) for various types of beams and various combinations of r and s. The values of b_i obtanied by Eq. (11) are presented in Table (1) for the first four eigenvalues and the Figs. (2 - 3) shows the graphics of the first and third modes of vibrations related to transverse displacement and Figs. (4 - 5) presents complete rotation of a clamped-clamped beam for a set of parameters r and s as indicated in Table (1). Note that the amplitude of the vibration mode decreases with increasing rotational inertia and shear deformation.

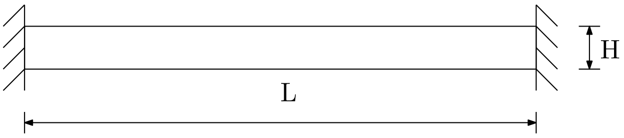

Figure 1. Clamped-clamped beam

Table 1. b_i eigenvalues for clamped-clamped Timoshenko beam

r	s	b_1	b_2	b_3	b_4
0	0	22.3733	61.6728	120.9034	199.8594
0.008	0.01399	22.2571	60.9390	118.3995	193.5505
0.04	0.0699	19.9419	48.8884	85.1489	125.5022
0.08	0.1399	15.7415	33.8419	54.8373	76.6488

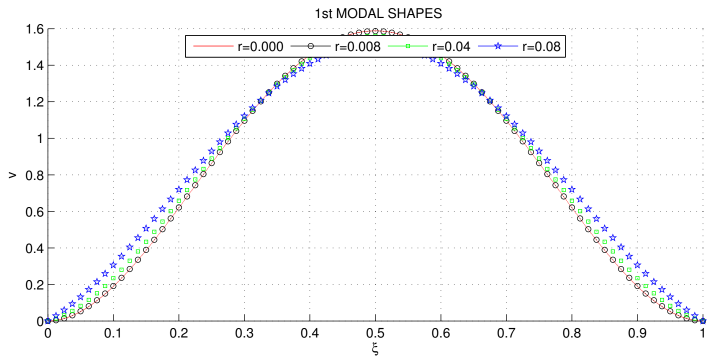

Figure 2. First modal shapes for clamped-clamped beam

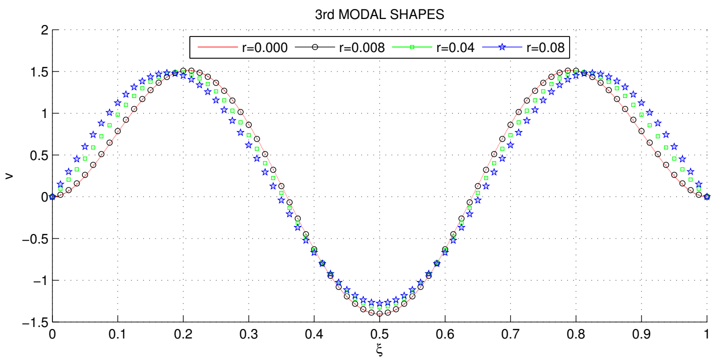

Figure 3. Third modal shapes for clamped-clamped beam

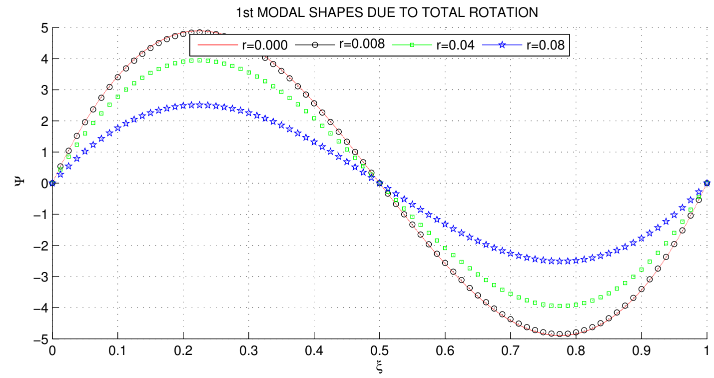

Figure 4. First modal shapes due to complete rotation

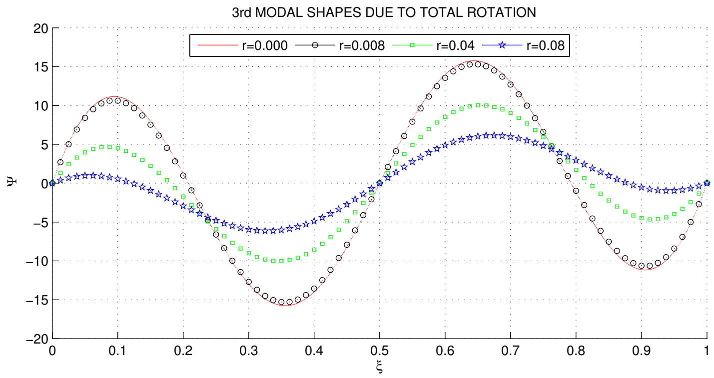

Figure 5. Third modal shapes due to complete rotation

Note that Timoshenko beam theory included the effect of rotatory inertia and transverse-shear deformation, and also these terms are isolated on coefficients r and s. When one or both of these constants are equal zero, it's obtained the equation of motion of previous beam theories. Euler-Bernoulli beam theory equation of motion can be expressed by develop of Eq. (9) with coefficients $r = s = 0$. Rayleigh and Shear beans theory can also be expressed with coefficients $s = 0$ and $r = 0$ respectively. The first four eigenvalues b_i for beams theories presented are showed in the Table (2).

Table 2. b_i eigenvalues for clamped-clamped beam

Beam Model	r	s	b_1	b_2	b_3	b_4
Euler-Bernoulli	0	0	22.3733	61.6728	120.9034	199.1768
Rayleigh	0.08	0	21.5399	54.1743	94.5919	137.9945
Shear	0	0.1399	15.9004	34.8350	56.8180	79.8545
Timoshenko	0.08	0.1399	15.7415	33.8419	54.8373	76.6488

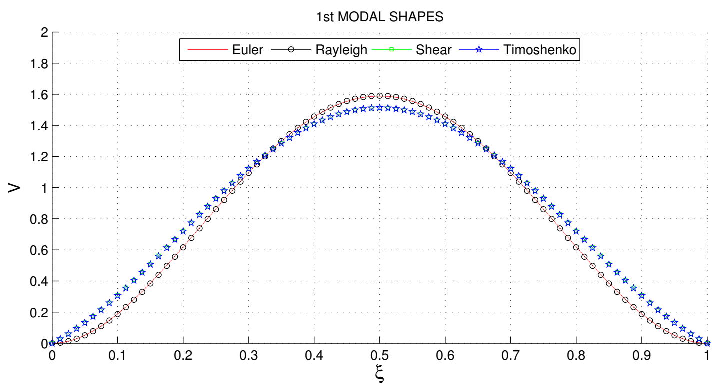

Figure 6. First modal shapes for clamped-clamped beam

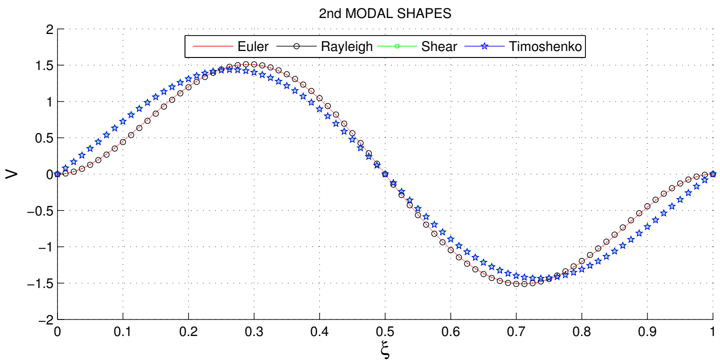

Figure 7. Second modal shapes for clamped-clamped beam

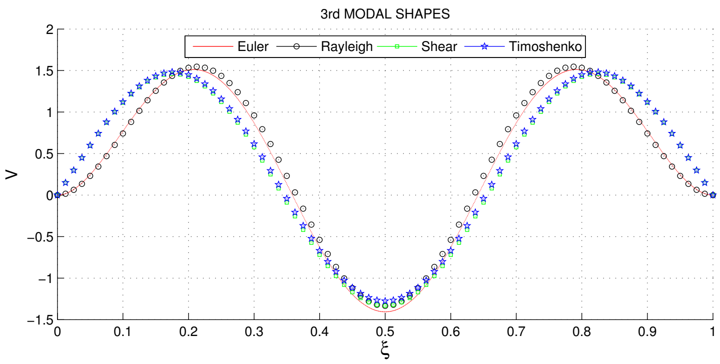

Figure 8. Third modal shapes for clamped-clamped beam

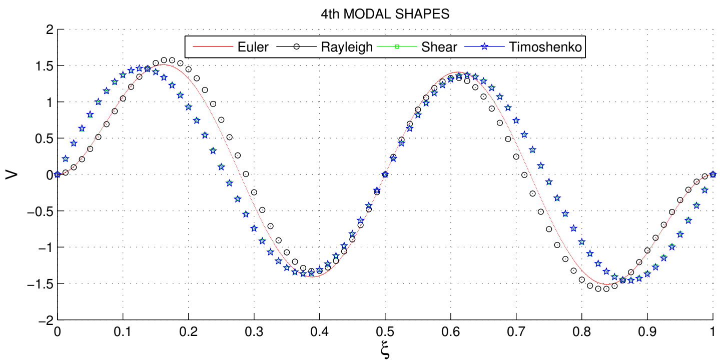

Figure 9. Fourth modal shapes for clamped-clamped beam

7.2 Clamped-free Beam

For a non-slender clamped-free beam showed in the Fig (10), with $H/L = 0.277128$, and $L = 1m$. The values of b_i obtained by Eq. (11) are presented in Table (3) for the first four eigenvalues some combinations of r and s. The Figs. (11 - 14) shows the graphics of the first four models of vibrations related to transverse displacement for a set of parameters r and s as indicated in Table (3). Note that the amplitude of the vibration mode decreases with increasing rotational inertia and shear deformation.

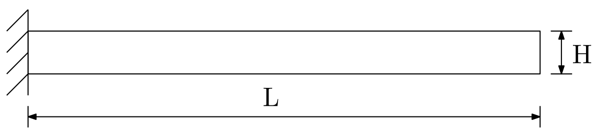

Figure 10. Clamped-free beam

Table 3. b_i eigenvalues for clamped-free beam

Beam Model	r	s	b_1	b_2	b_3	b_4
Euler-Bernoulli	0	0	3.5160	22.0345	61.6972	120.9019
Rayleigh	0.08	0	3.4648	20.0449	50.5283	87.7103
Shear	0	0.1399	3.3654	17.2235	39.3686	63.2530
Timoshenko	0.08	0.1399	3.3241	16.2897	36.7098	58.2826

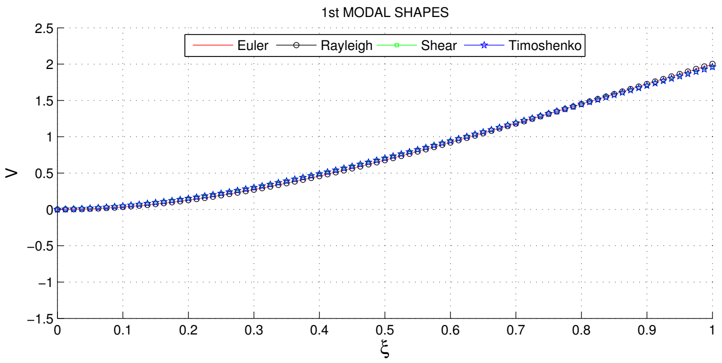

Figure 11. First modal shapes for clamped-free beam

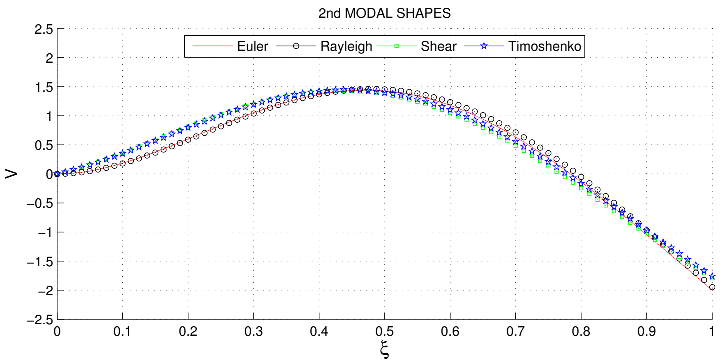

Figure 12. Second modal shapes for clamped-free beam

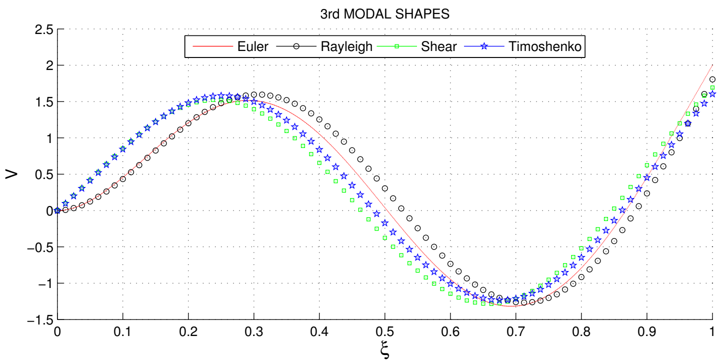

Figure 13. Third modal shapes for clamped-free beam

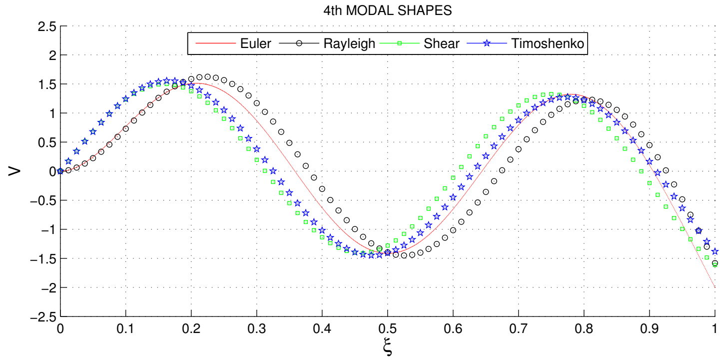

Figure 14. Fourth modal shapes for clamped-free beam

8. CONCLUSION

This paper, four approximate models for a transversely vibrating beam was presented: Euler-Bernoulli, Rayleigh, Shear and Timoshenko models. The equation of motion and the boundary conditions were obtained. For a given beam with r and s known the non-dimensional frequencies $b_i(i = 1, 2, 3, ...)$ can be found from the appropriate frequency equations or natural frequencies. The frequency equations are not simply due to their highly transcendental nature, but they can be solved numerically by using a Newton-Raphson root finding method. It was observed that the amplitude of the vibration mode decreases with increasing rotational inertia and shear deformation.

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